Our students record their designs in words to connect reading and writing to all that they do. They record their designs in numbers to transform the designs they have made into patterns.

We invented mathematics to record what we see. The universal symbols we use for recording allow us to compare data from seemingly dissimilar events so that we can see patterns and make connections. The common elements of the different ways to make five (squares, toothpicks, blocks) are the symbols we use to record them.

Concept, connecting, symbolic...

A fourth grade child has written as the answer for his homework.

$$\begin{array}{r} \mathbf{17} \\ \underline{\mathbf{+14}} \\ \mathbf{211} \end{array}$$

His father tries to help.

Father: Please count out seventeen paper clips and put them in a stack, then count out fourteen paper clips and put them in a different stack...
We write the amounts for the two different stacks like this.

$$\begin{array}{r} \mathbf{17} \\ \underline{\mathbf{+14}} \end{array}$$

Father: Now, how many paper clips do you have altogether?
Son: (After careful counting) Thirty-one.

$$\begin{array}{r} \mathbf{17} \\ \underline{\mathbf{+14}} \\ \mathbf{211} \end{array} \qquad\qquad \begin{array}{r} \mathbf{17} \\ \underline{\mathbf{+14}} \\ \mathbf{31} \end{array}$$

Father: How do you explain the two different answers?
Son: Thirty-one is for paper clips. Two hundred and eleven is for school.

Our students built each number from three to five or more with squares, toothpicks, blocks and cubes. They used materials to build the concept of the number in their minds.

Our students then record their designs using bits of paper or pencil lines. Recording gives our students an image of the design to save when the materials themselves are put away. These recordings are only shadowy images of the materials they represent. But they form the necessary connecting link between the materials and the universal symbols that follow. Universal symbols show that 2 + 3 = 5 for squares, for toothpicks, and for everything else as well.

The concept connected to the symbols. Concept, connecting, symbolic. Steps along the path to understanding. Concept, connecting, symbolic is not a one-way street. We travel up the steps, but steps go up and down. Symbolic, connecting, concept. The symbols connected to the concept, too.

Teacher: Here is a way we recorded a square design for three.

1 + 2

Teacher: How many different ways can you make square designs for these numbers? Your designs have to be one-plus-two designs.

(illustration 4-6-12)
(The number equation with several different interpretations along with it. Interpretations made with Power Block squares and with recordings of square designs.)

Our students record their designs using symbols. They use symbols to see what designs they can make. The answer for paper clips is the answer for school as well.

Three again...

Teacher: Today we will be doing three again. Here is a recording of a toothpick design for three.

(illustration 4-6-13)
(Toothpick design for three. Design in the shape of a triangle.)

Teacher: What does this look like to you?
Student: A tent.

The teacher writes "tent" next to the design.

Teacher: What else does this design look like to you?
Student: A triangle.
Teacher: When you begin looking at the recordings you made for your own toothpick designs, you will see many different things each design might be. But I want you to write down only one of the ways you see.

(illustration 4-6-14)
(Design book for toothpicks, not yet stapled together, showing word, design, word, design, word, design.)

Teacher: I am impressed with all the different words you have written to describe your toothpick designs. Now we will see if we can record your designs using numbers.
Please tell me what numbers you could use to describe this design.
Student: Two and one.
Teacher: Where is the two and where is the one? Please show me.

(illustration 4-6-15)
(Student breaking the design from the first toothpick illustration apart, with two toothpicks at the top and one toothpick at the bottom.)

Teacher: And how do we record two plus one in numbers?
Students: Write two plus one.
Teacher: Kim, shall we write it up and down or sideways?
Student: Up and down.

$$\begin{array}{r} \mathbf{2} \\ \mathbf{+1} \end{array}$$

Teacher: What number were we doing when we recorded this design?
Students: Three.
Teacher: And, what is two plus one?
Students: Three.
Teacher: Can anyone see a different way to describe this pattern with numbers?

Once our students understand the recording process, they record numbers for their designs.

Teacher: Here is a way we recorded a toothpick design for three.

$$\begin{array}{r} \mathbf{2} \\ \mathbf{+1} \end{array}$$

Teacher: How many different ways can you make toothpick designs for these numbers?

(illustration 4-6-16)
(The number equation with several different interpretations along with it. Interpretations made with toothpicks and with drawings of toothpicks.)

And again...
Teacher: Today we will be doing three again. Here is a recording of a design for three.

(illustration 4-6-17)
(Pattern Block design for three. One yellow piece with one green triangle sticking off the top left side and one green triangle sticking off the top right side.)

Teacher: What does this look like to you?
Student: A cat's face.

Teacher writes or stamps "cat face".

Teacher: What else does this design look like to you?
Student: A Halloween mask.
Teacher: Remember, when you write down words for your own designs, please write down only one of the ways you see your designs.

(illustration 4-6-18)
(Design book for Pattern Blocks, not yet stapled together, showing word, design, word, design, word, design. Also show design book stapled together, complete with the child's Xeroxed picture on the cover.)

Teacher: Now we will see if we can record your designs using numbers. Please tell me what numbers you could use to describe this design.
Student: Two and one.
Teacher: How do we record two plus one in numbers?

Once our students understand the recording process, they record the numbers for their designs.

Teacher: Here is a way we recorded a Pattern Block design for three.

$$\begin{array}{r} \mathbf{2} \\ \mathbf{+1} \end{array}$$

Teacher: How many different ways can you make Pattern Block designs for these numbers?

(illustration 4-6-19)
(The number equation with several different interpretations made with Pattern Blocks and with Pattern Block shapes glued to paper.)

Lesson Seven

Purpose	Learn the families of addition facts.
Summary	Students look for ways to make number combinations with Unifix Cubes using two different sets of rules.
Materials	Unifix Cubes, A-B cube paper.
Topic	The first set of rules leads to flash cards that go home.
Topic	Creating number combinations with Unifix Cubes and Rule One.
Topic	Creating flash cards to accompany Rule One.
Topic	Explore number patterns with Unifix Cubes and Rule Two.
Homework	We send the flash cards home.

Two colors...

Teacher: Please look in your tub of Unifix Cubes and pick out three cubes. For Unifix Cubes, each design is made by snapping the cubes together and standing them up.
The first rule for Unifix Cubes is that you can use no more than two colors. Once you decide which two colors you will use, you can use only those colors for your cube design.
Student: Can we use only one color if we want?
Teacher: After you make your first design, you may make designs using only one color. But I want your first three-cube design to have two colors in it so that I can tell which two colors you have chosen to use.
Please check your neighbor and have your neighbor check you to see if you each have three Unifix Cubes. Please also check to see that you each have chosen two different colors. You do not have to have the same colors as your neighbor.
The second rule is that the colors cannot be separated. If you are using reds and whites, the reds must all be touching each other and so must the whites. This is okay.

(illustration 4-7-1)
(Three cubes snapped together. Two reds and a white.)

Teacher: And this is not.

(illustration 4-7-2)
(Three cubes snapped together. Red, white, red.)

Teacher: Using the two rules, how many different designs can you make for three using your two different colors?

Each student's total is shared with everyone else in class. Ways are counted. Results are compared.

(illustration 4-7-3)
(All six ways shown with cubes.)

Teacher: Keep the ways you made three so we can look at them for patterns as we make other numbers. Using the same rule, how many ways can you find to make two using two colors?

(illustration 4-7-4)
(All four ways shown with cubes.)

Teacher: Keep the ways you made two and three. How many ways to make one?

(illustration 4-7-5)
(The two ways shown with cubes.)

Teacher: Lets see what we have so far.

Number Ways
1---------------------2
2---------------------4
3---------------------6
4---------------------

Teacher: Keep all the cube designs you have made so far. Using the same rule, how many ways do you think there might be to make four using two colors?

There may be students who see a pattern in the numbers and predict that there are eight ways. Prediction or not, the ways can be found.

(illustration 4-7-6)
(The eight ways shown with cubes.)

Number Ways
1---------------------2
2---------------------4
3---------------------6
4---------------------8
5---------------------

Teacher: What would your prediction be for the ways to make five?
Students: Ten.
Teacher: Can you see patterns in the cubes for the ways you made two? Could you see those patterns again in the ways you made three? Could you see patterns in the ways you made three that you could see again in the ways you made four? If you can find patterns, then maybe you can use these patterns to help you think of more ways to make five.

Looking does not guarantee seeing. Seeing comes with experience.

(illustration 4-7-7)
(Show the ways to make each succeeding number (one through five) side by side. Indicate in the caption that each succeeding way is made by duplicating the ways for the previous number and then adding one color to the bottom of one duplicate set and the other color to the bottom of the second duplicate set. Also show the staircases that some students use to see all the ways.)

Teacher: See if you can use the patterns you have found to make all the ways up to ten.

Flash cards...
Teacher: Lets record the Unifix Cube patterns you found using numbers.

(illustration 4-7-8)
(Unifix Cube design for three. One white cube and two red cubes.)

Teacher: The rule for cubes is that you have to describe the design from the top down. The cubes that are the same color are counted together. Tell me what numbers describe this design.
Student: One and two.
Teacher: And we record the one and two like this.

1
+2

Teacher: How would you record three whites and no reds?
Student: One plus one plus one.
Teacher: Remember, the rule is that cubes that are the same color are counted together.
Students: Three.
Teacher: Okay, three whites. How many reds?
Student: There aren't any.
Teacher: Zero reds. We write three whites and no reds like this.

3
+0

Teacher: How would we write three reds and no whites?
Student: Three plus zero.

3
+0

Teacher: Or like this, to tell it from the way with three whites and no reds.

0
+3

Teacher: Please write the numbers for all the ways for three.

(illustration 4-7-9)
(All the numbers written directly beneath the cubes.)

0	**1**	**2**	**2**	**1**	**3**
+3	**+2**	**+1**	**+1**	**+2**	**+0**

The numbers for making cubes from two colors are the numbers from which flash cards are made. When symbols are used, six ways now becomes four.

(illustration 4-7-10)
(Flash cards with duplicates eliminated.)

0	**1**	**2**	**3**
+3	**+2**	**+1**	**+0**

The families of facts are useful things to know. We add with more efficiency when we know our threes and fours and fives. So, we allow the use of calculators in our class, and we sometimes teach as if there were no calculators at all.

The ways to make twos and threes and fours with Unifix Cubes give each student the families of facts. Our students make the ways and record the ways on paper. We record the ways on flash cards as well.

(illustration 4-7-11)
(Flash cards made for the numbers 1 through 10. The accompanying text points out that there is a pattern to the number of 'facts' for each family. Show the flash cards laid out in the following pattern, so that the reader may see the pattern more clearly than is possible through just reading about what the pattern is supposed to be.)

Numbers for the illustration:

The two ways to add to 1	0 +1	1 +0		
The three ways to add to 2	0 +2	1 +1	2 +0	
The four ways to add to 3	0 +3	1 +2	2 +1	3 +0
The five ways to add to 4	0 +4	etc. etc.		

And on and on through ten (eleven ways)

We start with ones and twos and threes. We add in a new family of facts only when our students can say with confidence the numbers they have in hand. There is no reward for speed.

We need not be the ones to whom the cards are read. Students teach each other. The flash cards go home, as well. Brothers teach their sisters. Sisters teach their brothers. Parents teach them all.

New rule...

Teacher: The number we will start off with today is three. We still have the rule that you can use no more than two colors. The new rule is that the same colors do not have to be side-by-side. They can be separate.
Watch as I make some designs.

(illustration 4-7-12)
(Unifix Cubes stacks standing next to each other. Only two or three designs are made because there are not many possible combinations of ways to make three with the cubes. Make at least one stack using one color.)

Teacher: See how many different ways you can find to make Unifix Cubes designs for three Unifix Cubes. Remember, you can only use one or two colors and you cannot change your two colors once you pick them.
I need you all to help me check each other's designs. I need you to check three different things. First, check to see if each design uses only three cubes. Second, check to see if each design uses only one or two colors. Third, check to see if the design is standing up.
What three things did I want you to check?... First?... Second?... Third?...

Teacher: How many different designs for three can you make with your two different colors?
Student: Three.
Teacher: Show me your three ways.
Student: There are more ways.
Teacher: How many ways?
Student: Four.
Student: No, five.
Teacher: What I want to know is how many different ways you can find using just your two colors. But we are going to have a problem telling how many different designs you can make because you are not all using the same two colors. By my rule, these two might be the same design, even though they use different colors.

(illustration 4-7-13)
(Two designs that are the same except for color.)

Teacher: Because you are using different colors, we will have to use a way to describe your designs that will let us see if these designs are the same or not.

(illustration 4-7-14)
(Paper with a place for the A cube and a place for the B cube.)

Teacher: We will use the letters A and B to stand for your two different colors. Since using letters may be a little confusing at the start, I want you to work with a partner.

When you and your partner have chosen which two colors you will be using for your cube designs, put one of the colors in the A square and the other color in the B square.
Now, I am going to describe a design and I want to see if you and your partner can make it. I will always describe it from the top to the bottom.
The design is A-A-B. See if you and your partner can make an A-A-B design.

The teacher walks around the classroom checking to see if the teams of students understand how to make an A-A-B design. This is not the first time the students have described designs as A-A-B. Mathematics is patterns and connections. What we learn once we use again.

Teacher: How many different designs for three can you make with your two different colors?

(illustration 4-7-15)
(All the eight ways shown with cubes and written by the teacher in A, B format.)

For each of the materials used to make three in Lesson Four, imagination was the only limit to the number of ways that could be found. The limits imposed by the cubes require students to examine their own work and the work of their classmates in a different way.

Each student looks at the design he or she is making and compares it to each of the other designs she or he has already made. Each student's total is shared with everyone else in class. Ways are counted. Results are compared. Thinking is required. Colors do not matter. Only As and Bs.

If our students collectively find five different ways, then five is their answer. If they find six, then their answer is six. If they find eight ways and we think eight is all of the different ways they will find, we keep what we think to ourselves. We assign the problems. We teach the methods and the rules. We can say when a rule has not been followed, but we do not give answers. The answers come from the students themselves.

Teacher: Work together with your partner to see how many different ways you can find for two. Keep all the ways you have found for three.

(illustration 4-7-16)
(Students making and recording patterns for two.)

Teacher: How many ways did you find to make two?
Students: Four.
Teacher: How many ways do you think there are to make one using two different colors?
Students: (After some discussion) Two.

Number	**Ways**
1----------------------	**2**
2----------------------	**4**
3----------------------	**8**
4--------------------	**??**
5--------------------	

Teacher: I wonder how many ways there will be for four. Work with your partner to see how many ways you can find. Keep the ways you found for ones and twos and threes, unless you run out of cubes.

As the students work, the teacher walks around the classroom observing the approaches students use. Some students work in a seemingly random manner, trusting to luck to find all the ways. Other students look for patterns in what they are doing so they can make better sense out of their search.

Teacher: Can you see patterns in the ways you found to make two that you could see again in the ways you made three? If you can find patterns, then maybe you can use these patterns to help you think of more ways to make four.

Students may or may not make any connection between the ways to make two and three and the ways to make four. Looking does not guarantee seeing. Seeing comes with experience.

(illustration 4-7-17)
(Show the ways to make each succeeding number. The ways to make two, doubled, with the row added on top, become the ways to make three. Then the ways to make three, doubled with a row added on top

to become the ways to make four. The illustration will have captions explaining the process. The caption will also ask the question, "And how do we think we might make the fives?")

2, 4, 8, 16, 32. If making numbers with Unifix Cubes is the first time our students have seen the doubling pattern, they may not know that they can predict what the next numbers will be. The doubling pattern is the pattern for the powers of two (2^0, 2^1, 2^2, 2^3, 2^4, 2^5...).

Lesson Eight

Purpose	Learn to link number to area. Learn to prove answers found.
Summary	Students prove Power Block areas before creating shapes on their geoboards and proving areas of their created shapes. Adds to the wooden cube experiences from Lessons Four and Five. Provides more beginning number experiences for the older child.
Materials	Power Blocks, Geoboards and Blacklines for three levels of recording sheets. (One, four, twelve.)
Topic	Power Block S-1 square has an area of one, what are the areas of all the other shapes?
Topic	Geoboards—ways to make 2, without then with recording.
Topic	Reviewing other people's "2's".
Topic	Ways to make 3.
Topic	Ways to make 4 or more, while proving areas found.
Homework	If geoboards can be sent home, explorations are continued there.

Ryan...

Teacher: How many different shapes with an area of two can you make on your geoboards?

(illustration 4-8-1)
(The Rubber band forms a three sided hollow "square" around nails 3, 5, 6, 4. The band then goes from 4 to 1, to 2, to 3, forming an hour glass shape above the 3, 5, 6, 4, hollow square. Include a dotted line from point 3 to point 4 to make the square and the two triangles more visually apparent. Label this illustration "The shape that Ryan made".)

1. .2
3. .4
5. .6

Teacher: Ryan, you say the shape you have made has an area of two. Please prove your shape's area to me.
Ryan: See, there is a square at the bottom, and the two triangles add together to make another square. So that's two.
Teacher: What is the area of this shape?

(illustration 4-8-2)
(Classic two square unit rectangle made by stretching the rubber band around points 1, 3, 5, 6, 4, 2 and back to 1. Include a dotted line from point 3 to point 4 to make the two squares more visually apparent.)

1. .2
3. .4
5. .6

Ryan: Two.
Teacher: Prove it to me.
Ryan: Just count the squares inside. One, two.
Teacher: Can you prove it to me with the paper squares?
Ryan: Sure.

(illustration 4-8-3)
(Two paper squares placed inside the two unit geoboard rectangle from the illustration above.)

Teacher: Well, if the rectangle I just made has an area of two, how can the shape you made on your geoboard also have an area of two? Your shape is just my shape with two pieces taken out of it.

(illustration 4-8-4)
(Show Ryan's shape and the teacher's shape side by side on a geoboard.)

Ryan: Because it does. Look, there are two triangles here. And I can take the two triangles and put them together to make another square. So there is one square and another square and that's two. So the area is two.

(illustration 4-8-5)
(Demonstrate the steps Ryan used to prove the area was two. Put a paper square in Ryan's shape. Then put two paper half-squares (triangles) in the small triangles in Ryan's shape. The paper triangles stick out beyond the edges of the triangles in Ryan's geoboard shape. Then show the two paper triangles and one square fitted together to form a complete area of two square units.)

Ryan does not see that his shape cannot have the same area as the teacher's shape. He sees only that his shape is made up of a square and two triangles. Two triangles fit together to make another square. One square plus another is two. For Ryan, both shapes can have an area of two.

Hayley...

(illustration 4-8-6)
(The rubber band forms a right triangle around nails 1, 3, 5, 6 and back to 1. Label as "Hayley's triangle.")

1.
3.
5. .6

Teacher: What is the area of the triangle you have made? How many square units does it have?
Hayley: A half.
Teacher: How many square units in this shape?

(illustration 4-8-7)
(Classic two square unit rectangle.)

1. .2
3. .4
5. .6

Hayley: Two.
Teacher: Prove it with the paper squares.

Hayley places two paper squares inside the rectangle.

Teacher: What is half of two square units?
Hayley: One square unit.
Teacher: Now watch what happens when I divide the two square unit rectangle in half diagonally.

(illustration 4-8-8)
(The rectangle from the illustration above with a diagonal line drawn from point 1 to point 6.)

1. .2
3. .4
5. .6

Teacher: What is the area of each of the triangles that my diagonal dividing line has made?
Hayley: A half.
Teacher: A half-square unit? Is this triangle the same size as the triangle we got when we cut the paper square unit in half?
Hayley: Yes.
Teacher: Can you prove it with one of the triangle pieces of paper?

(illustration 4-8-9)
(Hayley lays a half-square triangle over the bottom of her larger triangle.)

Teacher: How does that prove that the triangle is a half-square unit?
Hayley: Because its a triangle.

Hayley does not hear or understand the *square unit* part of the teacher's question. Hayley sees the triangle she has made as half a rectangle. Hayley sees *all* triangles as halves.

Ways to make two...

Regardless of the grade we teach, our initial assessment is that no one knows anything. Our assessment means that we ask older students to find ways to make numbers at the start of the year, just as we ask younger ones.

Students in the earlier grades find ways to make numbers with squares or toothpicks or cubes. Older students find ways to make numbers using geoboards. We change the materials we use so that we can start our students at the beginning and not make them feel that they are repeating a lesson they might have had in an earlier year.

(illustration 4-8-10)
(Geoboard on the overhead with rubber band indicating one square unit.)

Teacher: The shape I have made on my geoboard has an area of one square unit. Area means how much space the shape covers. Please make this shape on your geoboard and then hold up your geoboard so I can see if I have made my instructions clear.

The students make one-square-unit shapes on their geoboards and hold up their boards.

Teacher: Now, see if you can make a shape on your geoboard that has an area of two square units.

(illustration 4-8-11)
(Show two different ways of making two square units correctly and one example of an attempt to make two that does not have an area of two. The two correct ways are both the classic two square rectangle way, one horizontal, one vertical. The way not to make two is the two-on-each-side square with an area of four.)

Power Blocks are a geoboard-readiness material. Students who have fit together blocks to make shapes have a frame of reference the teacher may use to define the meaning of two on a geoboard.

(illustration 4-8-12)
(Examples of Power Block shapes as they relate to proving simple areas of two on a geoboard.)

If no Power Blocks are available, paper squares may be used to demonstrate some areas.

Teacher: I can see that not everyone knows what I mean when I say "an area of two square units." I am going to give you some paper squares and triangles to use on your geoboards to help you understand what I mean when I say the shape has an area of two.

Paper squares and triangles are distributed.

Teacher: Please watch as I use my paper squares to see if the shapes I make on the overhead have an area of two.

(illustration 4-8-13)
(First picture: rectangular geoboard shape with an area of two. Second Picture: two squares of paper fit into the shape.)

Teacher: How many squares fit into this shape?
Students: Two.
Teacher: Then this shape has an area of two square units.

(illustration 4-8-14)
(First picture: Square geoboard shape with an area of four. Second Picture: Four squares of paper fit into the shape.)

Teacher: How many squares of paper fit into this shape?
Students: Four.
Teacher: Is this shape a way to make two?
Students: No.
Teacher: Okay, I want you to use your paper squares to help you find ways to make shapes on your geoboards with an area of two. Before you begin, though, I want to show you what the paper triangles are for.

(illustration 4-8-15)
(Paper square cut in two pieces diagonally. The illustration shows the original square uncut, then cut and pulled apart.)

Teacher: I made the triangles by cutting the squares in half diagonally. These triangles are called half square units. What does one-half square unit plus one-half square unit add up to?
Students: One.
Teacher: Make this shape on your geoboard and then use your squares and triangles to figure out its area.

(illustration 4-8-16)
(A trapezoid with an area of two. Triangle, square, triangle.)

Teacher: What is the area of this shape? Prove your answer with your paper squares and triangles.

If the students cannot find the area of the shape, the teacher finds it for them, and proves to all in class the area is two.

(illustration 4-8-17)
(Show the proof that the shape has an area of two. Put a paper square and two paper triangles in the shape, then take the paper out and show how the pieces add (or fit) together to make two.)

Teacher: See how many different shapes with an area of two square units you can make on your geoboard. For each new way you find, hold up your hand so I can come around and see what you have done. You will have to prove to me why you think the shape you have made has an area of two.

The students start creating and the teacher walks around checking their proofs.

Teacher: I am impressed with all the different ways to make two that you are creating, but you are creating shapes faster than I can come around to check them all. So I would like you to record each new way you discover so that you can keep on creating new ways while you are waiting for me to see your work.

Three levels of recording paper are used for copying geoboard designs. The first paper's recording space is the same size as the geoboard itself. The copy of the design is the same size as the design copied. This paper is used to help students learn how to reproduce their design. All students, even those who do not have perceptual difficulties, are given specific instructions in how to copy from the board to the paper.

(illustration 4-8-18)
(Geoboard design, with an arrow or finger pointing to the upper left hand corner. Blank recording sheet next to the geoboard. Match the geoboard shape to the dialog below. Identify the blackline number of the recording sheet.)

Teacher: How many nails down from the top does this shape begin? Count straight down from the top nail on your board to the first nail in the shape.
Students: Two.
Teacher: And how many nails over from the side? Count straight over from the side to the first nail in the shape.
Students: One.
Teacher. Count two dots down and one dot over on your recording sheet and mark that point. Please check your neighbor and have your neighbor check you to see that we all agree which point to mark.

(illustration 4-8-19)

(Same geoboard design, with an arrow or finger pointing to the lower left hand corner. The point described above is marked on the recording sheet.)

Teacher: How many nails straight up from the bottom does this shape begin? Count from the bottom to the first nail in the shape.
Students: One.
Teacher: And how many nails straight over from the side? Count from the side to the first nail in the shape.
Students: Three.
Teacher. Count one dot up and three dots over on your recording sheet and mark that point. Please check your neighbor and have your neighbor check you to see that we all agree which point to mark.
Now, connect the points.

(illustration 4-8-20)
(Recording sheet next to the geoboard with the first side drawn in.)

The process is repeated until the entire shape is drawn on the paper. Even though some students have no difficulty copying shapes from their geoboards to their papers, all students are given the instructions. When we teach all of our students, each of our students becomes a resource to every other student in our class.

Once we have explained how to record geoboard shapes, our students record the shapes they create on the large-size geoboard paper. Once students demonstrate that they can record accurately on the large geoboard paper, we decrease the size of the recording space.

(illustration 4-8-21)
(Geoboard designs recorded on four-geoboard paper. Identify the blackline number of the recording sheet.)

When students are comfortable with four to a page, we decrease the size of the recording space again.

(illustration 4-8-22)
(Geoboard designs recorded on twelve-geoboard paper. Identify the blackline number of the recording sheet.)

As our students are finding ways to make two, they are also learning how to copy accurately. In addition, they are learning to draw to scale, to add and subtract whole numbers and fractions, to calculate areas, to create and analyze geometric shapes, to think logically about what they are doing, and, most importantly, to prove the conclusions they may reach. They are learning there is more to doing numbers than just counting one, two, three.

Teacher: Think of all the ways you have found to make two. Now, think of new ways you have never thought of before. You may look at what others are doing and they may look at what you are doing as well.

Learning from the lesson...

The paper squares and triangles we give our students cannot prove all the ways there are to make shapes with an area of two. The squares and triangles are only tools. They are not the only tools we have. The problem-solving tool we need most is the ability to think.

Teacher: I was really impressed with all the ways you found to make shapes with an area of two square units yesterday. But yesterday was only our first day of learning about areas. We still have a lot of learning to do.
I am going to show you some of the shapes many of you made yesterday. Let's see if we can figure out their areas together.

We do our learning from the lesson. Learning does not mean we already have to know. Not every child has to understand today. As each child learns, another teacher is at our side. We can pair off students who do not understand, with those who do. If we find we cannot check everyone, we can ask students to prove to the person sitting next to them what they have done. If the shapes are too elaborate, we can add in the rule that the limit for shapes is what can be made with just one rubber band. If our students can record their work on the sheets of twelve, we can say, "Make twelve *twos* before you try a *three*. Each *two* must be unique." If the shapes are too complex, we can add the rule that each of the sides must end at a nail.

(illustration 4-8-23)
(Illustrate the rule for "each of the sides must end at a nail". Show some that do and some that do not.)

Ways to make three...

Teacher: Today, find ways to make shapes on your geoboard that have areas of three.
You may use the paper squares and triangles to help you prove your areas, but remember, the paper squares and triangles are not the only ways to prove the areas you create.
Please record each geoboard shape on your geoboard paper so that you can keep on creating while you are waiting for me to see your work. You do not need to raise your hand each time you create a shape because I will come around to everyone.
When you record your shapes, write the number for the area inside the shape you make. Soon, we will be finding areas for all kinds of different shapes. Writing the number inside will help you remember the area when I come around and ask you to prove it to me.

The students:
Create shapes that they believe have an area of three.
Record the shapes on geoboard paper.
Write the number for the area inside their recording for each shape.
Prove the areas for the shapes to the teacher as the teacher comes to each student's work space.

The teacher:
Moves around the room checking the shapes the students have created.
Provides reminders of recording techniques to those who have difficulty transferring geoboard designs to paper.
Helps students with proofs.
Looks for students who understand either how to record or how to prove, so that these students may help students who need assistance in recording or proving.
Looks for common areas of difficulty in proving the areas of shapes so that these areas of difficulty may be brought to the whole class for discussion.

Ways to make whatever we can...

As our students become comfortable with proving areas for twos and threes, we allow them to make shapes for whatever areas we feel comfortable helping them prove.

Teacher: Today, you may make shapes on your geoboard that have areas of two or three or four.
Student: Can we do five?
Teacher: Try a few fours, then ask me about five.

If we are using geoboards with our students for the very first time, we may wish to follow twos and threes with two and one-half then three and one-half. If we are more experienced, we may go through four and then allow our students to find the areas for any shape they can. How far we go depends on our own level of comfort and how quickly our students comprehend.

As we help our students learn to prove ways to make two and then three, we gradually gain knowledge of how to prove what has been done. As we increase the size of the shapes that may be made, our knowledge continues to expand.

We can outline what we know:
One rubber band is used to make each shape.
The ends of each side turn at a nail.
We look for the squares and triangles that together build the shape.
We look within the shape for what we know and what we do not recognize.
As we look, we ask ourselves to stop and think: Can we find the area from what we already know?
If we can, we add the areas for the smaller squares and triangles to find the total for the shape.
If we need more information, we surround the shape with a rectangle that encloses all the corners and the sides.
We look again for what we know and what we do not recognize.
As we look, we ask ourselves to stop and think again.

(illustration 4-8-24)
(Four shapes in parallel, carried through the steps listed immediately above. Not all shapes would be carried through all of the steps, since some shapes will be easier to prove. If the layout of the book permits, have the four proofs depicted literally side by side in four parallel columns. Use chalk on the

geoboards in one or two of the proofs, to show this as one of the possibilities available. Include a "chalk" comment in the caption.)

The two most common difficulties we encounter when learning how to prove are:

(illustration 4-8-25)
(The first difficulty is making sure that the rectangle placed around the shape includes all of the shape and nothing extra. Show a triangle surrounded by a rectangle, done correctly. Next to this, show the same triangle surrounded by a rectangle that is larger than is needed to enclose the triangle. Put a "yes" by the correct one and a "no" by the incorrect one, with a comment about the extra space enclosed by the rectangle.)

(The second difficulty is knowing what to do with the little extraneous piece of triangle that appears when a rectangle is put around one part of a shape, to calculate part of the external area, and the rectangle forms a small, irrelevant triangle for another part of the shape at the same time. This is hard to describe in words, but what to do with this extraneous and irrelevant bit of information gives teachers unfamiliar with geoboards much problem. Show a triangle with an arrow pointing to the extraneous bit of information. In the accompanying dialog, tell teachers why and how to ignore it.)

Summary

The assessments that we make...

In *Beginning Number*, we begin teaching addition, subtraction, fractions, geometry, measurement, problem solving, proofs and much, much more. In all the chapters that follow, learning numbers continues to be a part of all we do. *Beginning Number* starts a process of surrounding that continues all year long.

We use the activities in *Beginning Number* to surround our students with the concept of number. What assessment do we use to know when the surrounding is complete? What assessment do we use to know when it is time to move on?

The assessments that we make for our students are contained in everything we watch our students do. We see what they know in every design they create, in every shape they prove and in every recording they transcribe. The materials our students use let us see their thinking. We know what our students know just as parents know their infant's every word.

The assessments that we make for ourselves are:

How well are we able to move about the room paying attention to each child?
Have we involved every child in our lesson, regardless of present ability or past experience?
What new possibilities do we discover from our students?
What new ways do we discover for ourselves?
Can we prove things now that we could not prove before?
Is the environment we have established for our students one that surrounds each child with the concepts we want each child to learn?
Have we given learning the time it takes for learning to take place?
What would we do differently tomorrow? Can we look at our lessons and learn from each one how to make the next one even better?
Do we know that mathematics is not a series of lessons to be covered, it is a series of concepts to be explored?

I wasn't really reading it...

A conversation overheard:
Mother: (to 7-year-old son) I am very impressed with how well you read that story to me just now.
Son: No, you shouldn't be impressed because I wasn't really reading it.
Mother: What do you mean?
Son: I wasn't really reading it, because I already know all the words.
Mother: Well, what does it mean to read?
Son: Reading is like when you are in school and you try to say the words that you don't know and the teacher tells you that you got it wrong. Reading is when you don't know, not when you do.

Outside of school, we accept "Da da wee!" until "Go that way!" comes along. Inside of school, we circle the misspelled words and place red check marks by the numbers we call wrong. We point out on every paper we correct and in every way we can what each and every child cannot do. We focus on the wrongs and not the rights. Is it any wonder that this mother's son did not feel he could read? He had learned that reading was another of those things in school he could not do.

What happens to our belief in ourselves if learning emphasizes what we do not know? What happens if we are taught that we cannot, instead of that we can? In school we learn about all the things we cannot do. So now, we overlook our own creativity because we are not "creative." So now, we overlook our own intelligence because we are not "intelligent." So now, we overlook our own ability to learn mathematics because we are "no good at math."

The time has come for us to accept "da da wee" as an answer good enough for school. The time has come for us to teach our students what they can do and not what they cannot.

Questions from Teachers

1. Don't we have an obligation to move every child along as fast as that child is ready to move? Are not we doing a disservice to all children by having the slowest children be the determiners of the progress of the whole class?

We do not keep the quickest students from learning by keeping them longer at any one level. Learning is a continuous process. It is not tied to specific pages in a book or to a series of predetermined steps.

Does a person learning to play the piano learn less by playing the piano longer? Should we instead feel obligated to have the piano player who grasps techniques more quickly switch to saxophone, then trumpet, then drums? Or do we allow the learner to go into greater depth with just one instrument?

We do not short change our students by giving them the opportunity to explore concepts in depth. The measure of a child's success in school is not how many pages have been covered or how many calculations have been performed. The measure is the quality of the learning taking place.

What kind of people do we want the children who pass through our classrooms to become? What kind of adults do we feel our society needs?

Adults who are thinkers?
Adults who are doers?
Adults who share knowledge, not hoard it?
Adults who work well with people of all abilities?
Adults who take responsibility for others?

Or, do we need adults who feel success is measured by how far and how fast each individual alone can proceed. Shall we teach our students that no one else matters and there are no accomplishments to take pride in but one's own?

2. Do we really move everyone at once?

What happens if we do not? What happens to the child or the children we leave behind? When will they ever catch up? The most effective way to help a child who does not understand is to provide him or her with immediate and massive assistance. Assistance is immediately and massively available from one's own classmates. Where will this assistance come from if the classmates have moved on, leaving the slower child behind?

We want all of our students to learn. When the class moves from one number to the next, all move. When we have students who cannot do the number four, we pair those who cannot with those who can. When a child does not understand, we give that child more help, not less. We surround the child with the concept. We surround the child who does not understand with children who do.

The purpose of the number lessons in this chapter is to allow all of our students to experience three, then four, then five. Number is not the only purpose that we have. We also want all of our students to believe in themselves as learners. We do not separate children out. We teach all of our students that all of our students can learn.

3. Not all children are ready to explore the same number at the same time. Shouldn't we either assess each child to determine that child's appropriate starting number, or at least determine a range of numbers for each child to explore?

When we talk to an infant, do we speak only the words we feel the infant understands? We do not assess before we speak. Instead, we fill the infant's world with meaningful language. Arithmetic is less complicated than language. There is no need to assess our students for who should learn three or four or five. If we say to one child, "You must do four, you cannot do five," what are we teaching this child? Our students enter our rooms believing they can learn. We choose to keep this belief alive.

4. How do we know when to move on? How do we know when we have left no one behind?

We move from three to four when the majority of our class understands, which means teachers for three now fill our room. As the class grows in knowledge, there are more and more children to help the ones who need help. We surround the individual child with the means for learning and give each child the time.

We surround the child with the concept. At school. At home. Everywhere. The surrounding never stops, it takes place all year long. We make numbers a part of the child's world. Students do threes and fours and fives in *Beginning Number*. They do the numbers again in *Beginning Addition and Subtraction* and again when adding and subtracting in bases other than ten. Numbers are also a part of sorting, classifying, graphing, measuring and of everything else we do.

How do we know when the class understands? When our class knows, we will know that they know. We establish the environment. We provide the help. We allow the time. The class will learn what we want the class to know. We move on when we feel we should. If we find we have moved forward too soon, then we move back. We learn as our students do, through experience.

5. You indicate children should write numbers to record their designs. Do you mean kindergarten children as well?

Our students know what numbers are before they come to school. Written numbers exist all around them. They know printed words as well. Children who can pick up a crayon or a pencil can learn to make the connection between what they create with materials and the symbols we use to record these creations.

In kindergarten, children draw pictures that are symbols for trees, houses, family and friends. In many cases, they are already learning to print their names. On their own, they may be scribbling lines on paper as they pretend to write.

As long as we provide our students with the means for knowing the number to write, and as long as we accept whatever is written as good enough for now, we can give our students the opportunity to record whatever they see, whatever their age.

If we want kindergartners to write numbers, they can. If we want numbers to wait until the children are a year older, then waiting is what we do.

6. How do we keep track of where our students are, for ourselves, for the parents, for our administrators and for next year's teacher?

We know what our students know. We work with them every day. The challenge is not in keeping track, but in communicating what we know.

For the parents, we meet and explain what it is we are trying to do. The assignments we send home help parents to see the growth in their sons and daughters. At parent conferences, we provide updates on the progress of each parent's child.

For administrators, we fill in the forms we are asked to fill out. But we do not let the check lists that might be required become the focus of our teaching. If we were teaching language to an infant, we could report our progress in any way we were asked. But we would not let the means for reporting change how the language was taught.

For the next year's teacher, we provide anecdotal records, in which we report what each of our students can do. We stress the cans and not the cannots. How much use will next year's teacher make of any check list of skills that we might choose to pass on? Do we base what we teach in our classroom on a reading of every child's cumulative record? The next year's teacher may assume just like we, that no one knows anything, and start teaching at the start.