Chapter 10

Beginning Multiplication and Division

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Before We Begin

Confusion...

Teacher: There are twenty-four hours in a day. How many half hours are there in a day? When you have figured out an answer, please tell me what it is and how you got it. You may use your calculators to help you if you wish.

First student: Seven hundred twenty. Because twenty-four times thirty is seven hundred twenty.

Teacher: Where did you get the thirty?

First student: There are thirty minutes in half an hour.

Second student: Twelve. Because twenty-four divided by two is twelve.
Teacher: Where did you get the two?
Second student: There are two half hours in an hour.
Third student: Twenty-six. Because twenty-four and two is twenty-six.
Teacher: I guess I know where you got the two.
Fourth student: Twenty-seven. Because twenty-four times thirty is twenty-seven.
Teacher: Could you show me what you did on your calculator to get your answer?
Fourth student: Oops! I only put a three in my calculator instead of thirty. Twenty-four times thirty is fifty-four.

The fourth student had pushed the plus and not the times button on his calculator.

Fifth student: Eight. Because twenty-four divided by thirty is eight.
Teacher: Did you do the dividing on your calculator?
Fifth student: Yes.
Teacher: Please show me what you did and the answer you found.

The student shows the teacher the answer on his calculator.

Teacher: You see that dot in front of the eight?
Fifth student. Yes.
Teacher: That is a decimal point, like in dollars and cents. You read that number as eight-tenths or point eight.
Fifth student: Then there are eight-tenths half hours in a day.

There were students in this class who knew that twenty-four times two is forty-eight. There were students in this class who knew that twenty-four divided by a half also gives forty-eight. But very few students in this class understood how to use what they might know. Lessons in mathematics can teach us the algorithms we need to know; flash cards can put the facts inside our students' heads; calculators can perform the operations. Neither calculators nor flash cards nor algorithm rules can help us know the arithmetic to use.

Lesson One

Purpose	Learn what it means to multiply. Learn to search for patterns in multiplication arrays.
Summary	Students learn to create and record (individually and in matrices) multiplication problems. Matrices are searched for patterns.
Materials	Squares or tiles, blank 12x12 matrix blackline, Unifix Cubes, geoboards, multiplication wall matrix, lined paper.
Topic	Making and recording rectangles with squares.
Topic	Recording rectangle patterns on matrices.
Topic	Examining matrices for patterns.
Topic	Unifix matrices made and searched for patterns.
Topic	Geoboard matrices made and searched for patterns.
Topic	Add the matrix to the wall charts for pattern searches endlessly.
Topic	If needed, more matrices are made.
Homework	The search for patterns on the multiplication matrix can be shared at home.

Squares all in a row...

Multiplication is a record of the pattern for numbers added over and over again. Two plus two, plus two, plus two, plus two, gives us 2, 4, 6, 8, 10. Three plus three, plus three, plus three, plus three, gives us 3, 6, 9, 12, 15. Division is the record of a pattern as well. If mathematics were a language spoken at home, then our students would have learned to see patterns in mathematics before they came to school. But mathematics is not often spoken in most homes.

The "Beginning" in *Beginning Multiplication and Division* means the answers can be found by counting. "Beginning" also means beginning to understand.

Teacher: Today we will be making up multiplication problems and look for patterns in our answers. We'll start with fours. Please make a rectangle that is one square down and four squares across.

(illustration 10-1-1)
(One by four Power Block S-1 square rectangle.)

Teacher: How many squares altogether in this rectangle?
Students: Four.
Teacher: How many times do we have four?
Students: One.
Teacher: We would write that like this:

1 x 4 = 4

Teacher: Multiplication means how many times we have something. The x is the mathematical way of writing "how many times" or "times." We read this as "one times four equals four." It means we have four, one time.
Please add another row of four squares.

(illustration 10-1-2)
(Two by four square rectangle.)

Teacher: How many squares down?
Students: Two.
Teacher: How many squares across?
Students: Four.
Teacher: How many times do we have four in this rectangle?
Students: Two.
Teacher: What is the multiplication problem this rectangle represents?
Students: Four two times.
Teacher: We could say it that way, but the custom in mathematics is to put the times in the middle and say either two times four or four times two.
How many squares altogether?
Students: Eight.

1 x 4 = 4
2 x 4 = 8

Teacher: Add another row of four.

(illustration 10-1-3)
(Three by four square rectangle.)

Teacher: How many squares down?
Students: Three.
Teacher: How many squares across?
Students: Four.
Teacher: How many times do we have four in this rectangle?
Students: Three.
Teacher: What is the multiplication problem this rectangle represents?
Students: Three times four.
Teacher: How many squares altogether?
Students: Twelve.

1 x 4 = 4
2 x 4 = 8
3 x 4 = 12

Teacher: Add another row of four.

(illustration 10-1-4)
(Four by four square rectangle.)

1 x 4 = 4
2 x 4 = 8
3 x 4 = 12
4 x 4 = 16

Students add rows of four to the rectangle on the overhead until the teacher feels the students are ready to begin work on their own.

Teacher: I want you and your partner to make up and record multiplication problems for rectangles that are four squares across. Start just like we did on the overhead with a rectangle with only one row of four. Please make up and record a problem for each line on your recording paper. No fair skipping lines!
As you work, please look for any patterns you might see that you may share with the class.
After you record all the way to the bottom of your lined paper for rectangles with four squares across, you may record the multiplication problems for rectangles with five squares across. Start with a rectangle that has only one row of five in it and keep adding rows of five.

Students who finish fives may do sixes, sevens, eights, or any other number that they choose. There is no limit to the numbers students may explore. There is no limit to the patterns they may see.

At the end of the period, the whole class shares the patterns they have seen.

Making and recording rectangles for multiplication is a repeat of the Start with, go by activities in *Beginning Addition and Subtraction*, Lesson Two (page 129). The patterns are a repeat as well. The difference now is that we call the patterns that we see "multiplication." Multiplication is the record of patterns for numbers added over and over again.

Matrix...

Teacher: Today, we are going to record the multiplication rectangles in a different way. The way may be a little confusing at first, so I would like you and your partner to be sure to help everyone around you understand what I mean.
The squares on the twelve by twelve section of graph paper I have given you are the same size as the Power Block S-1 squares. Today we are going to make the rectangles right on the graph paper and record the answers right on the paper as well. What we will be making is called a matrix. The matrix will help us see even more patterns in the answers that we find.
The confusing part is knowing where to write the down and across and altogether numbers.

(illustration 10-1-5)
(Square rectangle four across and one down on the graph paper matrix. The rectangle is in the top row, with one edge touching the left edge of the matrix.)

Teacher: The first rule is that all the rectangles we make must start at the top of the matrix. Please put this rectangle on your paper. Please check your neighbors and have your neighbors check you to see that you all have the rectangle in the top row and starting at the left side.
How many squares across is this rectangle?
Students: Four.
Teacher: We write the number of squares across at the top of the paper, above the last square. How many down?
Students: One.
Teacher: We write the number of squares down to the left of the last square down. Since there is only one square down, we write it next to that square. How many squares in the rectangle altogether? What is one times four?
Students: Four.
Teacher: We write the four beneath the farthest square across and down.

(illustration 10-1-6)
(4 written outside the matrix above the fourth square over. This is a multiplication matrix. The 4 is written where the 4 would be written outside a multiplication matrix. 1 written outside the matrix beside the first square down. 4 written on the matrix beneath the square the farthest across and down. The square moved is shown slightly off center, so that it can be seen as moved but not yet gone.)

Teacher: (Showing a rectangle on the overhead that is four across and two down) Please put this rectangle on your paper. Please check your neighbors and have your neighbors check you to see that you all have the rectangle in the top two rows and starting at the left side.
How many squares across is this rectangle?
Students: Four.
Teacher: We write the number of squares across at the top of the paper, above the last square. There is already a four written there, so we don't have to write it again.
How many down?
Students: Two.

Teacher: We write the number of squares down to the left of the last square down. How many squares in the rectangle altogether? What is two times four?
Students: Eight.
Teacher: We write the eight beneath the farthest square across and down.

(illustration 10-1-7)
(2 written outside the matrix beside the second square down. 8 written on the matrix beneath the square the farthest across and down. The square moved is shown slightly off center, so that it can be seen as moved but not yet gone.)

Teacher: (Showing a rectangle on the overhead four across and three down) Please put this rectangle on your paper.
Please check your neighbors and have your neighbors check you to see that you all have the rectangle in the top three rows and starting at the left side.
How many squares across is this rectangle?
Students: Four.
Teacher: Where do we write the four?
Students: Four across at the top. Four is already written there.
Teacher: How many down?
Students: Three.
Teacher: Where do we write the number of squares down?
Students: To the left of the last square down.
Teacher: How many squares in the rectangle altogether? What is three times four?
Students: Twelve.
Teacher: We write the twelve beneath the farthest square across and down.

(illustration 10-1-8)
(3 written outside the matrix beside the second square down. 12 written on the matrix beneath the square the farthest across and down. The square moved is shown slightly off center, so that it can be seen as moved but not yet gone.)

Teacher: What size rectangle will I have you make next?

(illustration 10-1-9)
(The matrix with no squares on it and all the 'altogether' numbers clearly visible in a column.)

When the class has completed the matrix for rectangles with four across, the teacher starts a five-across rectangle.

Once the teacher feels the class understands the process, the students record their own information for the rectangles they may make on the matrix. The students work as the teacher wanders around the class observing how well students understand the process. As the students work, the teacher asks them to look for patterns that might help them know what numbers might come next. At the end of the period, the students share any patterns they have found with the whole class.

(illustration 10-1-10)
(Twelve by twelve multiplication matrix completely filled in. Row and column headings of 'across' and 'down' are written on the matrix.)

Teacher: I have placed a completely filled-in multiplication matrix on the overhead so that you can describe any patterns you have seen. It is okay if you and your partner have not yet completed your whole matrix. You can still share the patterns you have seen so far.
If you wish to describe a pattern to the rest of the class, it must be a pattern both you and your partner see. You and your partner must raise your hands together, so I will know you both see what you want to share. I may call on either one of you, so make sure you both see the pattern if you are both raising your hands.

If telling the whole class were the only way to share, how many students would have the chance to say aloud the patterns that they see? Communicating with one's partners lets more students practice using words to say what they have found.

Students may be so intent on sharing what they see with their partners, that they do not hear what is being shared by others in the class. If this is so, the teacher can give everyone in class four or five minutes to share with their workmates. When the partners finish sharing with each other, the only sharing allowed is that done with the whole class.

The students describe whatever patterns they see. They use whatever words they know to say what they mean. The teacher may help with words that might facilitate the descriptions: diagonal, parallel, perpendicular, horizontal, vertical, over one, down two, stair step, top to bottom, right to left, even, odd, goes up by one, goes down by two. Words are very much a part of math.

Teacher: Have you seen these patterns anywhere before?

The patterns the students see now have much in common with the Start with, go by patterns from *Beginning Addition and Subtraction*. The patterns also look much like the patterns students saw in the multiplication problems they did earlier in the lesson. Mathematics is patterns and connections. We can learn to make connections between the patterns that we see.

A different question to ask...

Teacher: How could you use this matrix to help you find answers to multiplication questions? If I asked you to tell me the answer to eight times seven, how could you use this matrix to know what the answer might be?

We may know that the matrix our students have constructed is a multiplication matrix and we may know how to use the matrix to find the answer to problems like two times three, but what we know is not always what our students know. The table was assembled from the answers to multiplication problems. But, the answers it stores will do our students no good if they do not know how to get back from the table the answers they have put in.

A new matrix...

(illustration 10-1-11)
(A blank twelve by twelve matrix with Tall written down the side and Number of sticks written across the top. Top left hand is written outside the top left-hand corner of the matrix. A pile of Unifix Cubes is next to the matrix.)

Teacher: Today we will make a new matrix. We will record patterns for sticks of Unifix Cubes to see what patterns we can find. Please make two separate sticks with four cubes in each stick.

(illustration 10-1-12)
(Two sticks of four cubes.)

Teacher: How many cubes tall is each stick?
Students: Four.
Teacher: Write the number of cubes tall along the side of the matrix with the word "Tall". We always start counting from the top left-hand corner of the matrix.
Please check your neighbor and have your neighbor check you to see if you have written the four on your matrix in the same place I have written the four on mine.
What is the number of sticks?
Students: Two.
Teacher: We write the number of sticks along the side of the matrix with the words "Number of sticks". We always start counting from the top left-hand corner of the matrix.
Please check your neighbor and have your neighbor check you to see if you have written the two on your matrix in the same place I have written the two on mine.
How many cubes do we have altogether?
Students: Eight.
Teacher: I have two sticks of four cubes. That means I have four, two times. Two times four is eight. Watch where I put the eight on the matrix.
The hard part of recording multiplication answers on a matrix is knowing where to write the number. One way to tell that the number is in the right place is to draw a line straight across from the four and then draw a line straight down from the two. The square where the two lines cross is the square where we put the answer for how many cubes we have altogether.

(illustration 10-1-13)
(Twelve by twelve matrix with a 4 written four spaces down on the left side, a 2 written one space over on the top and an 8 written inside the matrix, four spaces down and two spaces over, with light lines drawn from the two and the four intersecting at the square with 8 inside it. Two sticks of four Unifix Cubes next to the matrix.)

Teacher: Please check each other's work very carefully to see that we all recorded the total number of cubes in the same place.
Add one cube to each of your two sticks. How many cubes tall are your sticks now?
Students: Five.

Teacher: Where do we write the number for how tall?
Students: Along the Tall side of the matrix.
Teacher: Where along the Tall side?
Students: Counting down from the top left-hand corner of the matrix.
Teacher: What is the number of sticks?
Students: Two.
Teacher: Where do we write the number of sticks?
Students: Along the Number of sticks side of the matrix. We already have a two there.
Teacher: How many cubes do we have altogether?
Students: Ten.
Teacher: I have two sticks of five cubes. That's five, two times. Two times five is ten. Where do I put the ten on the matrix?
Student: Below the eight.
Teacher: Let's check and see.

(illustration 10-1-14)
(Twelve by twelve matrix from above illustration with a 5 written five spaces down on the left side, and with light lines drawn from the two and the five intersecting at the square with 10 inside it. Two sticks of five Unifix Cubes next to the matrix.)

Teacher: What size sticks will I have you make next?
Students: Six.

When the class has completed the matrix for two sticks, the teacher starts the process again with three. Once the teacher feels the class understands the process, the students record their own information for the Unifix Cube sticks on the matrix. The students work as the teacher wanders around the class observing how well students understand the process of recording the cube sticks on the matrix. As the students work, they look for patterns that might help them know what numbers might come next.

At the end of the period, the students share with the whole class any patterns they have found.

(illustration 10-1-15)
(Multiplication matrix completely filled in. The headings of Number of sticks and Tall are added to the matrix.)

Teacher: I have placed a completely filled-in multiplication matrix on the overhead so that you can point out any patterns you have seen. It is okay if you and your partner have not yet completed your whole matrix. You can still share the patterns you have seen so far.
If you wish to describe a pattern to the rest of the class, it must be a pattern both you and your partner see. You and your partner must raise your hands together, so I will know you both see what you want to share.
Have you seen these patterns before?

A matrix made with cubes has much in common with a matrix made with squares. Mathematics is patterns and connections. Looking for patterns helps us make connections between the patterns we see now and the patterns we have seen before.

Making connections...

Our students first saw the patterns for multiplication as they explored Start with, go bys, though they may not have known that the patterns they saw had anything to do with multiplication. The patterns appear again as our students work with squares and then Unifix Cubes. Each new matrix shows the patterns that the materials have in common. For each new matrix we ask:

What patterns do you see?
Have you seen these patterns before?

Our students create matrices with squares and cubes. If we wish our students to see these numbers once or twice again, we can use beans and cups, Pattern Blocks, or students carrying books. We call what our students are doing "multiplication" so they learn to match the word "multiplication" with the patterns that they see. Our students keep on making matrices until they see that the matrix they are making now is one they have made before.

Next to the 0-99...

Teacher: Look at this multiplication matrix and tell me what patterns you see.

(illustration 10-1-16)

(Show a multiplication matrix next to the 0-99 matrix from the Patterns and Connections Chapter. The matrix has "top right, top left, bottom right, bottom left" written on it for reference.)

Patterns are everywhere. In *Patterns and Connections* (page 026) we used the 0-99 matrix to give our students the opportunity to look at numbers for the patterns they reveal. Now, we place the multiplication matrix next to the 0-99 matrix on our wall.

Students look today and say what they can see. They look again in a week or a month or a year and say what more they see. Numbers record patterns. The more we know of numbers, the more of their wonder we see.

Lesson Two

Purpose	Learn to create multiplication and division problems, with and without remainders.
Summary	Students create and solve problems in a sideways L and answer three basic questions that we ask.
Materials	Squares, sideways L, chalkboards, paper.
Topic	Create problems, ignore remainder. Create problems for each other. Create problems, remainder not ignored. Remainder recorded as a fraction.
Topic	Create times problems with hands full of squares, remainders are ignored.
Topic	Answer three questions for the sideways L, no remainders yet.
Topic	Remainders added in.
Topic	Three questions asked again.
Topic	Remainders recorded as fractions.
Homework	Once the process is understood, our students create multiplication problems at home.

Rectangles...

Teacher: Today we are going to learn how to use a sideways L to make up multiplication and division problems with rectangles of squares. First, I take a handful of squares.
Let's count my handful of squares. Seventeen squares in my hand. I am going to use these squares to make a rectangle. I'll make my rectangle three squares across.

(illustration 10-2-1)
(Seventeen Power Block squares on the overhead inside a large sideways L. "Sideways L" is the descriptive term used here to designate the traditional symbol or shape drawn down the left and across the top of numbers to be divided. Fifteen squares in a 3 x 5 rectangle inside the sideways L. The top edge of the rectangle is along the top edge of the sideways L. The left edge of the rectangle is along the side of the sideways L. The two leftover squares are outside the sideways L.)

Teacher: How many rows down did I make with my squares?
Student: Five.
Student: Six. You have some left that you can put in row six.
Teacher: If I put these two extra squares in my rectangle, it won't be a rectangle any more. Later on, we'll learn what to do with any extra squares, but for now I'll just push them aside.
How many squares are down the side of my rectangle?
Students: Five.
Teacher: How many squares are across the top of my rectangle?
Students: Three.
Teacher: How many squares altogether in my rectangle? What is five times three?
Students: Fifteen.
Teacher: Here's how I record five times three equals fifteen using the sideways L.

$$5\overline{\smash{)}15}^{\;3}$$

Teacher: I put the number of squares across on the top of my sideways L. I put the number of squares down on the side. I put the number of squares altogether inside.

There are other ways to record multiplication problems. For example:

$$\begin{array}{r} 3 \\ \underline{\times 5} \\ 15 \end{array} \quad \text{or} \quad 3 \times 5 = 15$$

Recording in the sideways L helps our students see a connection: Fifteen is the answer to the question, "What is five times three?" What is the question that has five as the answer? Or three?

Teacher: I am going to take a new handful of squares and make up another problem. Twenty-one squares are in my hand. I think I'll make the rectangle four across this time.

(illustration 10-2-2)
(Twenty-one squares on the overhead in a 4 x 5 rectangle. The rectangle is correctly positioned inside the sideways L. The one square left is outside the L.)

Teacher: How many rows down?
Student: Five.
Teacher: What did I do with the leftover square?
Students: Pushed it aside.
Teacher: The problems we make up don't always have to have leftover squares. It just depends on how many squares we have. How many squares altogether in my rectangle? What is four times five?
Students: Twenty.
Teacher: Where do I put the twenty?
Students: Inside the sideways L.
Teacher: Where do I put the number of squares across?
Students: On the top.
Teacher: Where do I put the number of squares down?
Students: On the side.

$$\begin{array}{r|r} & \underline{4} \\ 5 & 20 \end{array}$$

Chalkboards for assessment...

Teacher: I am going to take a new handful of squares and make up another problem. This time, I want you to write the steps for recording the problem on your individual chalkboards, so I can tell how well I've done at teaching you so far today. Twenty-three squares are in my hand. How many squares across should I say are in the rectangle I make?
Student: Three.

(illustration 10-2-3)
(Twenty-three squares on the overhead in a 3 x 7 rectangle. Two squares left.)

Teacher: How many rows down did I make with my squares?
Student: Seven.
Teacher: What do I do with the leftovers?
Students: Push them aside.
Teacher: How many squares altogether in my rectangle? What is three times seven?
Students: Twenty-one.
Teacher: Show me on your chalkboards where I put the twenty-one. Where do I put the number of squares across? Where do I put the number of squares down?

$$\begin{array}{r|r} & \underline{3} \\ 7 & 21 \end{array}$$

The chalkboards are an assessment tool for knowing when our students are ready to begin working on their own. We see what they understand from what they write. After we have them write the numbers for the problems we create, we have them make up problems on their own and hold up the answers they record. We know from the boards which students need our help first.

When we see that everyone is ready to begin, we give our students paper for recording all their work.

A different kind of problem...

Teacher: I want you and your workmate to solve this problem together. I have a rectangle that has twenty-eight squares in it. It has four squares across the top. How many squares does it have down the side?

$$? \overline{)28}^{\ 4}$$

Teacher: When you and your workmate have an answer, write it on your individual chalkboard and put your board face down on your desk. When everyone has an answer, we'll hold up the boards and see what answers you have found.

Students form the rectangle that has twenty-eight squares altogether and four squares across.

Teacher: I see that the answer most commonly on your boards is seven. Let's see if the squares agree with what you say.

(illustration 10-2-4)
(A four by seven rectangle in the sideways L.)

The teacher puts the rectangle on the overhead to demonstrate the meaning of the question. The students who understood will have their understanding verified. The students who did not comprehend the meaning of the question, can now see what it was they were trying to find out.

The teacher gives more problems for squares across the top and squares altogether. The chalkboards help assess how many students understand and how many do not.

Teacher: Here's a different kind of problem. I have a rectangle that has thirty-five squares in it. It has seven squares down the side. How many squares does it have across the top?

$$7 \overline{)35}^{\ ?}$$

Teacher: When you and your workmate have an answer, write it on your individual chalkboard and put your board face down on your desk. When everyone has an answer, we'll hold up the boards and see what answers you have found.

Connected...

Multiplication and division are connected. Multiplication means adding the same number over and over again. We can find the answer for 3 x 5 by adding 3 + 3 + 3 + 3 + 3, or adding 5 + 5 + 5. The answer tells us how many altogether. Division asks what numbers did we add.

We ask three questions for the rectangles we make with the squares:

1) We know the squares across and the squares down. How many total squares?

$$d \overline{)t?}^{\ a}$$

2) We know the total squares and the squares across. How many squares down?

$$d? \overline{)t}^{\ a}$$

3) We know the total squares and the squares down. How many squares across?

$$d \overline{)t}^{\ a?}$$

Nothing pushed aside...

Teacher: How many squares do I have in this handful?
Students: Twenty-six.

Teacher: How many rows across should I make?
Student: Six.

(illustration 10-2-5)
(6 x 4 rectangle inside the sideways L. Two leftover squares.)

Teacher: Show me on your individual chalkboards how I should record this problem.

$$\begin{array}{r|l} & \underline{\ 4} \\ 6 & 24 \end{array}$$

Teacher: This is how we learned to record the problem when we pushed aside the squares that did not fit. Now we'll learn what to do when we don't push the extra squares aside.
How many squares did I have altogether in my hand?
Students: Twenty-six.
Teacher: How many squares across and down?
Students: Four across and six down.
Teacher: How many leftover squares?
Students: Two.
Teacher: In mathematics, the custom is to write the number of leftover squares alongside the number of squares across, like this.

$$\begin{array}{r|l} & \underline{\ 4}\ r\ 2 \\ 6 & 26 \end{array}$$

Teacher: What do you think the r stands for?
Student: Remainder.

If no one knows what the r stands for, the teacher volunteers the information.

Teacher: I am going to take a new handful of squares and make them into a rectangle. This time, I want you to record the problem on your individual chalkboard, so I can see if I have succeeded in teaching you how to record remainders.

The teacher gives examples for recording with remainders as the students demonstrate their level of understanding on their chalkboards.

After students write the numbers for the problems we create, they make up problems on their own and hold up the answers they record. We see what they understand from what they write. When we see that everyone is ready to begin, we give our students paper for recording their work.

Students make handsful of squares into rectangles. Some will have remainders, some will not. Either way, nothing is pushed aside.

Three questions asked again...

Teacher: I want you and your workmate to solve this problem together. I have a rectangle that has twenty-eight squares in it and two left over. It has four squares across the top. How many squares does it have down the side? When everyone has an answer, we'll hold up the boards and see what answers you have found. Don't forget to record the two left over.

The students work together to form the rectangle that has twenty-eight squares and two leftover.

Teacher: I see that seven is the answer most commonly on your boards. Let's see if the squares agree with what you say.

(illustration 10-2-6)
(A four by seven rectangle with two squares left over.)

The teacher puts the rectangle and the two leftover squares on the overhead to demonstrate the meaning of the question. The thinking involved in deciding how many squares are down for four across with two leftover is much more complicated than just knowing what rectangle to make. The chalkboards help us know how many students understand.

We ask three questions for the rectangles:

1) We know the squares across, the squares down and the leftover squares. How many total squares?

a r l
d | t?

2) We know the total squares, the squares across and the leftover squares. How many squares down?

a r l
d? | t

3) We know the total squares and the squares down. How many squares across and how many leftover squares?

a? r l?
d | t

High school remainders...

Teacher: Today, I am going to show you a different way to record remainders. In elementary school, we teach people to write the number of leftover squares by putting an r next to the number leftover as you have already been doing. But when you get to high school, you'll write remainders differently. Here is a problem we have already worked out.

4 r 2
7 | 30

Teacher: In high school, instead of writing remainder two, you take the remainder and write it over the number of squares down.

$$7 \,|\, 30 \quad 4\tfrac{2}{7}$$

Teacher: We read this remainder as *two over seven* or *two-sevenths.*

(illustration 10-2-7)
(A 7 x 4 rectangle with two squares along the right side of the rectangle, in effect two-sevenths of a row.)

Teacher: *Two* is the number of squares remaining in our handful after we have made four columns of seven squares each. *Seven* is the number in each column down. We have *two* of the *seven* (or two-sevenths) of the squares needed to make a new column. From now on, you may record your remainders the way you will be required to in high school.

When students record the leftover squares as fractions and not just as numbers, they see fractions as a part of the numbers they use. Our students know about fractions. They even have an idea of what the numbers mean. As our students learn more about fractions, they will learn more of the meaning that leftovers have. We surround our students with mathematics. Writing remainders as fractions is a part of the surrounding that we do.

Questions we might ask...

As our students multiply and divide with squares we might ask:

How many different rectangles can you make with twenty-four squares? No remainders allowed.
How many ways can you make rectangles for seventeen squares? No remainders allowed.
Which number of squares between one and fifty makes the most rectangles with no remainders?
Which number of squares makes the least?
Keep track of all the ways you try for each new number. What patterns can you see? Have you seen these patterns before?
Will there be more ways to make rectangles with no remainders for twenty-four or for twenty-five?
What else can you find out?

Lesson Three

Purpose	Learn to look for non numeric patterns. Connect the non numeric patterns to numeric patterns seen before.
Summary	Students create patterns with Unifix Cubes, break the patterns apart and study the "break-aparts" for patterns. Students connect the cube patterns to patterns in multiplication and on the 0-99 chart.
Materials	Unifix Cubes, Crayons, blacklines for recording break-aparts, 0-99 and multiplication matrices.
Topic	Break-apart pattern searches.
Topic	Look again at the matrices on the wall for patterns.

Unifix break-aparts...

(illustration 10-3-1)
(One A-A-A-B stack of Unifix Cubes.)

Teacher: I have made a Unifix Cube stick for an A-A-A-B pattern. I would like you and your partner to make several A-A-A-B sticks. Your colors do not have to be the same as anyone else's, but please use the same two colors for all of the sticks you make. Please also make sure that the B color is at the bottom end of all the sticks you make.
Now watch as I make a train from my sticks. It is very important that we all make our trains the same way. We will be looking for patterns, and the only way we can compare the patterns we see is to start with the same kind of train.

(illustration 10-3-2)
(Illustrate the making of the train.)

Teacher: Please have your neighbor check you and you check your neighbor to see that we have all made our trains the same way. The train should have the A end, or the bumpy end, of the stick to your right. The train should be an A-A-A-B, A-A-A-B, A-A-A-B train.

The teacher walks around the room to make sure everyone has the same train heading to the right.

Teacher: Now I am going to give you a rule for breaking your trains apart. As you follow the rule, see what patterns you can find.
The first way we will break the train is by threes. Watch as I break three cubes off the front of my train and slide them up. Three, then slide up. Three, then slide up. Three, then slide up. The bumpy end always faces to the right.

(Illustration 10-3-3)
(Breaking and sliding of the Unifix Cubes. The illustration must make clear what "sliding" means.)

Teacher: Please break your train the same way. Be very careful to break only three cubes off the front each time and slide them up. Please check your neighbor and have your neighbor check you to see that I have made my instructions clear.

The teacher walks around the room to see if the instructions are understood. Although patterns can be seen no matter which way the cubes are broken and no matter which way the students slide the cubes, unless everyone breaks and slides the same way, there is no common basis for comparison.

Teacher: What patterns do you think you see? Before you tell me what pattern you wish to share, please tell it to your partner. You both must see it before you both raise your hands. I will only call on teams with both hands raised.

(illustration 10-3-4)
(One train broken apart by the rule to show the pattern to be seen.)

Student: The As make an upside-down staircase.
Student: Then they make a right-side-up staircase, then upside-down again, then right-side-up.
Student: The Bs all go in a slanty line.
Teacher: The slanty line is called a diagonal. Which way is the diagonal going?
Student: Down.
Teacher: I'll write top-left, top-right, bottom-left, bottom-right as we did for the 0-99 matrix.
Student: Top right to bottom left. Then it skips a row and starts all over again going down.

Teacher: What other patterns do you see?

The students continue to describe the patterns that they see.

Teacher: I want you to put your train back together again, but I want you to leave just enough of the break-aparts we did for three so you can still see the patterns you described.
Now let's break your train apart by fives and see what patterns we can find. Remember, we always break from the right, or the bumpy end, and slide up what we break.

(illustration 10-3-5)
(A break apart for fives.)

Teacher: What patterns do you see? Remember, to share your pattern with your partner first.

The students share the patterns they see with each other and with the class. When there are no more patterns to be shared, the students save a part of what they have made and reassemble their train.

Teacher: Now let's break it apart by fours and see what patterns we can find.

(illustration 10-3-6)
(A break apart for fours.)

Teacher: What other ways can we break apart the cubes? What other patterns will you see?

More A-A-A-B cubes may be added to the trains. When a student's starting set of A-A-A-B cubes runs low, different colors may be used to create the A-A-A-B pattern. A-A-A-B represents the order of the colors used. Red, red, red, white and black, black, black, blue are equally A-A-A-B.

When cubes are in short supply, students use crayons and graph paper to record their patterns.

(illustration 10-3-7)
(A, A, A, B patterns recorded on graph paper. The cubes are next to the graph paper recording, so it can be seen that the recording is a copy of a cube design.)

Teacher: I wonder if there is a pattern we might see in all your break-aparts that will tell us the next pattern we will see before we break the train apart. Look at just the B cubes in each break-apart and see what patterns you can see.

(illustration 10-3-8)
(Break aparts for three, four, five, six, seven, eight, nine.)

Teacher: Look at your break-apart for threes and tell me what the B cubes are doing.
Student: A diagonal going down.
Teacher: I will record that with an arrow going down. Look at your break-apart for fours and tell me what the B cubes are doing.
Student: Going in a line straight down.
Teacher: What are the B cubes doing for fives?
Student: They are a diagonal going down the other way.
Teacher: For sixes?
Student: A checkerboard.
Teacher: For sevens?

(illustration 10-3-9)
(The break-apart recording sheet has two rows of squares. The top row is called "Break" and has the numbers 2, 3, 4, 5, 6, 7, 8, ...through 20 written in the squares. The bottom row is called "Pattern" and is where the 'B' pattern is recorded. A diagonal arrow is drawn in the 3 space. A down arrow drawn in the 4 space. A diagonal arrow drawn in the 5 space. A little checkerboard drawn in the 6 space. Fill in for seven and beyond.)

Pattern for cubes can be described many ways. They can be recorded many ways as well. It does not matter how the patterns are described or recorded as long as the descriptions and recordings are consistently applied. If "diagonal going down" is used to describe the break-aparts for three, then "diagonal going down" is used to describe the break-aparts for seven. If a checkerboard design is written as the way for sixes, then a checkerboard design is written for tens as well.

When the students have explored the break-aparts for A-A-A-B patterns, they explore the break aparts for A-A-A-A-B patterns, then A-A-A-A-A-B or A-A-B patterns and so on. As students continue to explore, we ask them to predict the patterns they might find. There is no limit to the questions we might ask. There is no limit to the patterns our students might see.

The matrices on the wall...

(illustration 10-3-10)
(The multiplication matrix and the 0-99 matrix side-by-side.)

We have asked our students to look at these matrices before, and now we ask them to look again.

Teacher: Look at this multiplication matrix and tell me what patterns you see.
Do you see patterns here like the patterns we saw in the break-aparts?
Do you see diagonal patterns going down to the left or to the right?
Do you see checkerboard patterns?
Are there patterns in small groups of squares?

(illustration 10-3-11)
Show a pattern for a cluster of squares to illustrate what this last question is asking. For example, take the digits in any group of four adjacent numbers (in the same 2 x 2 square) on a multiplication matrix, add the numbers that are catty-corner from each other and note that the resultant answers are always one digit apart.)

Teacher: What other patterns might there be that you have not seen before?
What would a multiplication matrix look like if we colored in every third or fourth or fifth square?
What patterns can you see in the 0-99 matrix on the wall?
What would the 0-99 matrix look like if you colored in boxes counting by twos or threes?

Mathematics is patterns and connections. Numbers are not the only way we have to record the patterns that we see.

Lesson Four

Purpose	Learn the multiplication number facts.
Summary	Students use flash cards to commit number facts to memory and examine a matrix for patterns to reduce the number of facts to be learned from 100 to 36.
Materials	Blackline for flash cards, shaded 10 x 10 multiplication matrix.
Topic	Flash cards are used to put the multiplication facts in each student's head.
Homework	Flash cards are sent home.

The families of facts...

Families of facts are useful things to know. It is good for our students to know the answer to three times four without having to enter the numbers in a calculator each time. The families of facts we want our students to learn come from the multiplication matrices they have made. We use these cards in the same ways we used the cards for addition or subtraction:

(illustration 10-4-1)
(Sets of flash cards grouped by patterns.)

Students working alone or with partners can read through their stack of cards. Answers they guess correctly go in pile number one. Answers not quite right go in pile number two. Students then review pile number two. Can any cards from pile two now be placed in pile one? There is no time limit to this activity.
Students who wish to compete with one another may face off with a set of cards to see who can say the most right answers without looking at the chart.
Students take the cards home for practice or for teaching younger siblings what they know.

Not 100 separate ones....

The flash cards help our students commit the families of facts to memory, but do they need 100 cards to learn the 100 facts from 1 through 10? They do not present a challenge to anybody's memory. The ones and tens together are 36 (not 40) of the 100 facts. How many of the remaining 64 facts are unique? Just 36.

Proof:

(illustration 10-4-2)
(10 x10 matrix with the multiples of one and ten shaded and all duplicate multiples shaded, as well. "Duplicate" means 3 x 4 is not shaded, but 4 x 3 is. The non-shaded numbers create the following staircase design within the full matrix.)

1	2	3	4	5	6	7	8	9	10
2	4	6	8	10	12	14	16	18	20
3	6	9	12	15	18	21	24	27	30
4	8	12	16	20	24	28	32	36	40
5	10	15	20	25	30	35	40	45	50
6	12	18	24	30	36	42	48	54	60
7	14	21	28	35	42	49	56	63	70
8	16	24	32	40	48	56	64	72	80
9	18	27	36	45	54	63	72	81	90
10	20	30	40	50	60	70	80	90	100
	8	+7	+6	+5	+4	+3	+2	+1	=36

The stair-step pattern for the 36 facts shows that there are less numbers to memorize when there are patterns seen. Less effort is required to memorize 36 related facts than to remember 100 separate ones. The pattern for the families of facts our students need to know is:

2x2	3x3	4x4	5x5	6x6	7x7	8x8	9x9
2x3	3x4	4x5	5x6	6x7	7x8	8x9	
2x4	3x5	4x6	5x7	6x8	7x9		
2x5	3x6	4x7	5x8	6x9			
2x6	3x7	4x8	5x9				
2x7	3x8	4x9					
2x8	3x9						
2x9							

Flash cards are a convenient teaching tool, but flash cards do not make the numbers a part of our students lives. Patterns may reduce the number of facts to memorize, but the best way to commit a fact to memory is to use it in a meaningful way. Our students learn the facts more permanently as we make the facts a part of their lives.

Lesson Five

Purpose	Learn to think about what the numbers in multiplication and division problems represent.
Summary	We create multiplication and division problems that our students solve, as they identify what the numbers in their answers represent.
Materials	Students and objects in our room, chalkboards, blackline for a 12 x 12 blank matrix (optional).
Topic	Multiplication word problems created with people and objects in the room.
Topic	The problems can be recorded in matrix form.
Topic	Division word problems created with people and objects in the room.
Topic	The problems can be recorded in matrix form.
Homework	The process of creation can go home.

Books from the shelf...

Teacher: Zachary, Jacob, Beth and Colleen, will you each please bring three books from the bookshelf in the back of the room and come stand by my desk.
How many people are standing by my desk?
Students: Four.

4 children

Teacher: How many books is each person holding?
Students: Three.

4 children
3 books

Teacher: What problem is this?
Students: Three times four.

If our students do not know that three books for each of the four children is multiplication, we teach them that what they learned for squares works for books as well.

Teacher: What is the answer to the problem?
Students: Twelve.
Teacher: Twelve what?
Students: Twelve books.

4 children
<u>x3 books</u>
12 books

Our students know that three times four is twelve. They learned the answer from their squares. Three times four with squares did not ask our students to think about what each number meant. All the numbers stood for squares—squares across, squares down and total squares. We write the words alongside the numbers for children and for books so our students have an opportunity to think about what the numbers mean. Three times four is twelve. But twelve what? Twelve children? Twelve books? Or, twelve of something else?

In Lesson Two, our students used the sideways L to record the multiplication problems they created with squares. Now, they record the problems in a different way. There is more than one way to record. The method we use should be the one most suitable for the problem at hand.

Teacher: You can see how I recorded the problem I made up. This time, I want you to use your individual chalkboards to record the problem I create.
I do not want you to tell me the answers to my questions out loud. I want you to write the problem and the answer on your chalkboard. You may work with a partner if you wish. When you have written the problem and its answer, please turn your chalkboard face down on your desk. Please make sure you record the words that go with the numbers, so I can tell what you think the numbers mean.
Each person at this first table please get four books from the shelf and then stand by your chair. Everyone else, record on your boards the problem for the people and the books.
Okay, everybody hold up your chalkboards so I may see what you have written.
The answer I see most often on your boards is sixteen books. So the class answer to this problem is sixteen books. If you do not have sixteen books on your board, please have someone who has the class answer on their board show you how they got sixteen.

The structure of the lesson parallels that used in *Beginning Addition and Subtraction*, Lesson Six (page 139). The teacher makes up a problem using the children in the room and the objects they carry. The students write the numbers, words and answers on their chalkboards and turn their boards face down on their desks until everyone in class is ready. The boards are held up and the teacher announces the class answer, including the words that give meaning to the numbers in the answer.

A matrix if we choose...

If we so choose, we may have our students organize the numbers for the books and people in a matrix.

(illustration 10-5-1)
(A twelve by twelve multiplication matrix. The side is labeled *children*. The top is labeled books.)

The top is labeled Books. The side is labeled Children. What label should we give the numbers in the middle? The label can be Books that children have. Numbers have meaning. We help our students think what meaning numbers have.

If our students create a matrix for classmates carrying books, we ask the class together to see what patterns they may find. The patterns for the books are the same patterns our students have seen so many times before. Mathematics is patterns and connections. Patterns make connections between squares and cubes and people with books. Patterns show us that the numbers we use to record different events might not be so different after all.

Division, too…

Teacher: Anthony please bring twenty books to the front of the room.
How many books did Anthony bring to the front of the room?
Students: Twenty.

20 books

Teacher: Robin, Richard, Trisha and Jack, please come up and see if you can distribute the books equally among yourselves.

We create problems with people and materials for multiplication. We create problems with people and materials for division, too.

Teacher: How many people did I have come up to share the books?
Students: Four.

20 books
4 children

Teacher: How many books does each person have?
Students: Five.

20 books
<u>÷4 children</u>
5 books for each child

Since we recorded the multiplication problems with the numbers in a column, we record division in a column as well. There is nothing magical about recording division in a sideways L.

Teacher: Paul please bring twenty-three books to the front of the room.
How many books did Paul bring to the front of the room?
Students: Twenty-three.

23 books

Teacher: Jean, Joan, Jim and Bill, please come up and distribute the books equally among yourselves. If there are any extra, set them to one side.
How many people did I have come up to share the books?
Students: Four.

23 books
4 children

Teacher: How many books does each person have?
Students: Five.

23 books
<u>÷4 children</u>
5 books for each child

Teacher: How many books are left?
Students: Three.

23 books
<u>÷4 children</u>
5 books for each child, r 3

Teacher: What does r 3 mean?
Students: Remainder of three.
Teacher: Three what?
Students: Three books.

We can write the remainder as 3, or we can ask our students to write the remainder as a fraction. Is it 3/4 or is it 3/5? Do we know which way to write the remainder? The denominator tells us how many people were trying to share, so the remainder is 3/4.

Teacher: You can see how I have recorded the problems I made up for the books at the front of the room. Now I want you to use your chalkboards to record a problem I create.
If I have a stack of twenty-five books at the front of the room and I ask six children to divide the books equally among themselves, how many books will each child have? Will there be any books leftover?
Write the problem and the answer on your chalkboard. You may work with a partner if you wish. When you have written the problem and its answer, please turn your chalkboard face down on your desk. Please make sure you record the words that go with the numbers, so I can tell what you think the numbers mean.
Okay, everybody hold up your chalkboards so I may see what you have written.

The teacher makes up stories about stacks of books and children to divide the books. The students write numbers, words and answers for the stories on their chalkboards. The boards are held up and the teacher announces the class answer, including the words that give meaning to the numbers.

A matrix of a different kind...

Teacher: Let's see what patterns we might find if we record the answers to the problems with the books in a matrix of a different kind.

(illustration 10-5-2)
(Blank matrix on the overhead. The top is labeled books in the stack. The side is labeled children. The top and the side are numbered from 1 through 12.)

Teacher: Let's start with a stack of three books. How many books would each child have if only one child had all the books?
Students: Three books.
Teacher: There are still three books in our stack. How many books would each child have if two children shared all the books and each child had to have an equal number?
Students: One book each, with a leftover book.

If our students do not know how many books each child would have and how many books would be left, they act out the problem with three books and two children. The answer already exists in the materials we use. We use the materials to find the answers that we need.

Teacher: How many books would each child have if three children shared all the books and each child had to have an equal number?
Students: One book each.
Teacher: When we reach the point where each child has one book and there aren't any books left, we start with a new stack and see what answers we find.
Now, let's start with a stack of four books. How many books would each child have if only one child had all the books?
Students: Four books.
Teacher: How many books would each child have if two children shared all the books?
Students: Two books each.
Teacher: How many books if three children shared?
Students: One book each with one leftover book.
Teacher: How many books if four children shared?
Students: One book each.
Teacher: What do we do when we reach the point where each child has one book and there aren't any books left?
Students: Start with a new stack.

We can create the matrix at the overhead or let our students work on their own. When we or they are finished, we look at the numbers and describe the patterns that we see.

(illustration 10-5-3)
(Division matrix filled in.)

We continue to create problems for our students to solve for addition, subtraction, multiplication and division. We also create problems for fractions, decimals, percents and anything else that might come along. The inspirations we use for the problems we invent are the students and the materials in our class and everyday objects that our students have at home. Mathematics at school. Mathematics at home. Mathematics wherever we or they go.

Lesson Six

Purpose	Learn to create stories for multiplication and division problems. Learn to see the stories in numbers everywhere.
Summary	Students write or draw stories for numbers we provide. Students look for number stories in their own lives.
Materials	Writing and drawing paper.
Topic	Students write stories or draw illustrations to accompany multiplication and division problems.
Topic	Selected stories written one day are read as creative inspiration the next.
Topic	Students write stories or draw illustrations to accompany multiplication and division problems that they provide.
Topic	If we choose to, we select some student stories as problems for the class to solve.
Homework	Numbers found at home can be brought to school to share.

More stories for numbers...

Teacher: What is the answer to this problem?

$$\begin{array}{r} 4 \\ \underline{\times 5} \end{array}$$

Students: Twenty.
Teacher: Who would like to tell me a story to go with these numbers? Jason?
Jason: There were four elephants playing in the jungle. Each elephant was carrying five children. So there were twenty children riding on the elephants.
Teacher: Does Jason's story fit the numbers?
Some students: Yes.
Teacher: We could act out Jason's story, but I am going to draw a picture for it instead, to see if it fits the numbers.

(illustration 10-6-1)
(A line drawing of the four elephants and the twenty children. Add numbers next to the drawings.)

Teacher: How many elephants?
Students: Four.
Teacher: How many children on each elephant?
Students: Five.
Teacher: How many children altogether?
Students: Twenty.
Teacher: Okay, then Jason's story fits the problem I wrote on the overhead. Another story?
Brenda: I had four kittens. Sarah had five kittens. So, altogether, we had twenty kittens.
Teacher: Does that story fit the numbers?
Some students: Yes.
Some students: No.
Teacher: I'll draw a picture of what Brenda said to see if it fits the numbers.

(illustration 10-6-2)
(A stick figure drawing of two girls, one holding four kittens, one holding five kittens. Add numbers next to the drawings.)

Teacher: Brenda and Sarah do not really look like the girls in my drawing. This is just how I draw people. How many kittens is the first person in my drawing carrying?
Students: Four.
Teacher: How many kittens is the second person in my drawing carrying?
Students: Five.
Teacher: How many kittens are there altogether?
Students: Nine.
Teacher: Does Brenda's story match the numbers in my problem?
Students: No.

Teacher: How could we change Brenda's story so that it does match?
Student: Have more girls to carry the kittens.
Teacher: How many girls should I have?
Student: Have four.
Teacher: So, I now have four girls carrying the nine kittens?
Student: No. Have the four girls each carrying five kittens.
Teacher: I'll draw that and see what it looks like.

(illustration 10-6-3)
(Stick figure drawing of four girls carrying five kittens each. Write numbers next to the figures in the drawing.)

Teacher: How many people have I drawn?
Students: Four.
Teacher: How many kittens is each person holding?
Students: Five.
Teacher: How many kittens altogether?
Students: Twenty.
Teacher: Does Brenda's story fit the problem now?
Students: Yes.

4	**7**	**3**	**9**
×5	**×8**	**×6**	**×2**

Teacher: I want each of you to write a different story for each problem I have written on the overhead. If you do not know how to spell a word, bring me your spelling notebook and I will write the word for you. See how different you can make your story from anyone else's.

As our students write, we add words to their spelling notebooks as needed. Students using the Reading Program may either write or stamp their stories. At the end of the writing time, the teacher collects the stories and reads the best examples of creative story writing to the whole class before the lesson is repeated on another day. We add selected samples of the writings to each student's portfolio.

The stories students write form a connection between numbers on paper and numbers in life. Our goal is that all the students in our class can show through writing or through drawing that they know what numbers mean.

Division stories, too...

Teacher: What is the answer to this problem?

10 | 20

Students: Two.
Teacher: Who would like to tell me a story to go with these numbers?

We ask our students to create stories for division just as they did for multiplication.

From our own lives...

Teacher: You have already written stories for me from your own lives for addition and subtraction. Today I want you to write me some stories from your own lives for multiplication. I will give you an example from my life to show you what I mean.
During football season, I go to games on Saturday afternoons with my father and my brothers. We take turns bringing the food for our tailgate party. Last Saturday, it was my turn to bring the food. I knew there would be my father, my three brothers, their wives or girlfriends and five children. That was fifteen people, counting me. I had to bring enough bread so that each person could make two sandwiches.
How many pieces of bread are in most sandwiches?
Students: Two.
Teacher: So, how many pieces of bread would I have to bring for each person?
Students: Four.
Teacher: Then the problem I had was how many pieces of bread would I have to bring for fifteen people. What was fifteen times four?
My story is a multiplication problem because I would have to multiply fifteen times four to find out how much bread to bring. But it would have been a division problem if I had a loaf of bread that had forty slices in it and I had to figure out how many people could share that loaf.

Today, I want you to write multiplication or division stories from your own lives. You do not have to be able to figure out the answer to the story you tell. Figuring out the answers is something we can all do together later on. You may use your spelling notebooks if you need help with the spelling of a word.

If our students need help in finding stories, we look again at the list we used for addition and subtraction:

Birthday parties, pizza parties, journeys to the ice cream store.
Playing baseball, basketball, football, stickball, hockey, soccer, volleyball, and any other sport.
Dodgeball, four square, hopscotch, jump rope, jacks or pick-up sticks, tag, or hide-and-seek.
Monopoly, Trivial Pursuit, Go to the Head of the Class, Clue, Parcheesi, Yahtzee.
Ice skating, dancing, swimming and the water slides.
Part-time jobs baby-sitting or pulling weeds from the neighbor's lawn.
Allowances, spending, saving, shopping at the grocery store or mall.
Arcade games or Nintendo games at home.
Playing with friends, visiting relatives, sitting home alone.
Bedtimes, meal times, favorite times, worst times, show-and-tell times, sitting still times.
Sad times, glad times, bored times, overjoyed times.
Picnics, camping, hiking, visiting the park. Swings and slides and Jungle Gyms.
Flying kites, making models, playing dolls, playing war.
Flying in a plane, riding in a bus, train, or subway, traveling in a car.
Vacations to Disney World or Disneyland or Opreland.
Adventures in the snow, water, or sand.
Circuses, fairs, carnivals, pageants and parades.
Aunts, uncles, fathers, mothers, cousins, siblings, relatives of all kinds.
Doctors, nurses, dentists, broken bones, scrapes and fevers, numbers of cavities to fill.
Weather wet or dry. Lightning storms, hail storms, hurricanes and tornadoes.
Pets, big and little, lost and found.
Favorite movies or TV shows.
Reading books in libraries or at home, comics, or the sides of cereal boxes.
Moving from one city to another or from one house to the next.
Visiting friends in hospitals or in jail.
Time and weight and measure. Growing every week and month and year.
How long to travel from there to here.

As we ask our students to write, we encourage creativity. We add the best examples of their writing to their portfolios. If we choose to, we select some student stories as problems for the class to solve. We present selected stories to the class, and groups of students find and share solutions to the problems posed.

Lesson Seven

Purpose	Learn that problems to be solved are everywhere around.
Summary	Students seek the multiplication and division problems that already exist in their lives.
Materials	Everything around.
Topic	Students describe number situations that exist. Finding the answers to the situations is not required yet.
Topic	Selected problems from the situations found are solved by the class.
Homework	The search for number situations extends to home.

Problems posed…

Our students can already add and subtract. They can already multiply and divide as well. The structure for addition and subtraction is mirrored in the structure for multiplication and division. Counting gives the easy answers. Calculators can be used to find the rest. Our students think about the arithmetic they need to use. They work together to find solutions to the problems posed. Each learns to write about the problems and justify the answers found. Reasonableness is expected. The class votes on answers and defends them to students whose votes are not the same.

We take problems from the real world. We want our students to solve problems in situations that surround them wherever they go. We connect mathematics to the lives of our students. Mathematics is more than what we learn in school.

Looking at the windows...

We used windows for sorting and classifying to focus the wandering attention of our students. We said, "Look *at* the windows you are looking *through*!" We use windows again to show our students that they are surrounded by multiplication and division everywhere they go.

Teacher: How many rows of windows are going across the side wall of our classroom?
Students: Four.
Teacher: How many windows are in each row?
Students: Five.
Teacher: How many windows altogether?
Students: Twenty.

(illustration 10-7-1)
(Drawing of the twenty windows, with 4 x 5 = 20 written next to the drawing.)

Teacher: This is what one of the windows at my house looks like.

(illustration 10-7-2)
(Two squares by five squares drawn on the overhead.)

Teacher: What multiplication problem does my window at home represent?
Students: Two times five.
Teacher: How many windows altogether?
Students: Ten.
Teacher: We write that as:

$$2 \times 5 = 10$$

Your assignment for homework tonight is to find as many window patterns for multiplication as you can. Draw the windows. Write the multiplication problem the windows represent next to your drawing. Also, write where you saw the windows that you drew.

The windows we ask our students to see are everywhere around. When we make looking at the widows into looking at multiplication, we teach our students to look for mathematics everywhere.

Looking at everything around...

Teacher: Your assignment for today is to find as many patterns for multiplication as you can. Draw what you find. Next to your drawing write the multiplication problem your drawing represents. Also, write where you found the problem you have drawn.

Examples we may give our students for finding multiplication patterns in and out of school:

Boxes or cans on grocery store shelves.
Eggs in cartons, oranges in crates, flowers in bunches.
Tiles on walls, floors and ceilings.
Cement slabs on sidewalks and streets.
Boards for houses and fences.
Patterns on wallpaper.
Tapes or CDs in the record-store racks.
Seats in the auditorium.
Members of the marching band.
Bottles on the water truck.
Trees in the Christmas tree lot.
Number of fingers in class.
Parking spaces in the city lot.
Estimates for the number of children attending our school.
Lunch money collected each day. Each week. Each month. For the year.
Seconds in minutes. Minutes in hours. Hours in days, weeks, or years.
Heart beats in a minute. How many in a day, week, or year?
Number of breaths in a day.
Rows, columns, stacks and lines.
Any number grouped in a regular way.

We ask our students to find the multiplication and division problems that surround them wherever they go. Mathematics is simple and basic and straightforward. Mathematics is everywhere around.

Summary

First, however...

Teacher: Today we will be working with the attribute blocks. I want your group to put your set of blocks all in a line, so that each block in the line is only one attribute different than the block next to it. I will give you an example of what I mean on the overhead.

First, however, I want you to take your group's attribute blocks out of the box and share the blocks with the members of your group so that you each have an equal number to start with.

The attribute lesson will eventually be about sorting and classifying. But before the students create their line of blocks, we ask them to do the division that is required.

As our students share the blocks among their group we may ask them:

How many blocks altogether?
How many children in your group?
How many blocks did you give to each child?
Were there any blocks leftover?
If there were, what did you do with these blocks?
Write a description (or draw a picture) of how you shared.

We present mathematics in school in the same natural and meaningful way as is the language we learn at home. We make mathematics a language spoken at school. We see the mathematics in situations whenever the situations arise. Opportunities for multiplication and division are already a common part of our world. We see beyond the lesson at hand, whatever lesson it might be, to see the mathematics that is already there. Learning is natural. The problems we wish to create are waiting for us everywhere.

Questions from Teachers

1. Some of the multiplication and division problems the children encounter as they look for problems in their own environment may be very, very large. Should we delay trying to find answers to the larger problems until we are teaching advanced multiplication and division?

Finding the answer to how many tiles altogether in a wall 72 tiles high and 102 tiles wide may be more than our students care to count, but the understanding involved is no different than for a wall 3 tiles high and 4 tiles wide.

Even though we can require the problems to be simpler if we wish, we cannot comprehend the grandeur of the elephant by studying the elephant one square inch at a time. Our students are capable of understanding complexity. Complexity is a part of all our lives. If we segment learning into too many small pieces in the name of simplicity, we can make the learning harder than we know.

The materials we use give our students the opportunity to understand. Multiplication is not a set of flash cards to be memorized. It is a way of describing patterns that we see. We let our students show us for themselves if we think a problem is too hard for them to do.

2. Shouldn't we put more emphasis on memorizing the facts? Knowing all the facts will help our students pass the tests they face in school. Knowing all the facts will help our students find the answers faster when they leave our room.

The facts are already in calculators. Students will have calculators when they leave our rooms.

Do we equate learning with memorizing facts? Do we believe that if we simply have our students memorize well enough, that learning will appear? We can teach our students to get the right answer without thinking, but this kind of mindless thinking is what a calculator does. Shall we ignore the existence of the tools we have and turn our students into tools instead? Or, shall we use the tools we have to help our students learn?

Building better horse-drawn carriages did not keep Model Ts from coming off the Ford assembly line. The world changes in spite of any desire we might have to cling to the ways that were. We can ready our students for the world they will face, or we can pretend that the old ways were the best.

Calculators take away the need to memorize, but calculators cannot think. Thinking is the most important skill our students can take with them from school.

Quotes from the past:

> Students today can't prepare bark to calculate their problems. They depend upon their slates which are more expensive. What will they do when their slate is dropped and it breaks? They will be unable to write!
>
> Teachers Conference, 1703

> Students today depend upon paper too much. They don't know how to write on slate without getting chalk dust all over themselves. They can't clean a slate properly. What will they do when they run out of paper?
>
> Principal's Association, 1815

> Students today depend too much on ink. They don't know how to use a pen knife to sharpen a quill. Pen and ink will never replace the quill.
>
> National Association of Teachers, 1907

> Students today depend upon store bought ink. They don't know how to make their own. When they run out of ink they will be unable to write words or ciphers until their next trip to the settlement. This is a sad commentary on modern education.
>
> The Rural American Teacher, 1929

> Students today depend upon these expensive fountain pens. They can no longer write with a straight pen and nib. We parents must not allow them to wallow in such luxury to the detriment of learning how to cope in the real business world, which is not so extravagant.
>
> P. T. A. Gazette, 1941

> Ball point pens will be the ruin of education in our country. Students use these devices and then throw them away. The American virtues of thrift and frugality are being discarded. Businesses and banks will never allow such expensive luxuries.
>
> Federal Teacher, 1950

> Students today depend too much on hand-held calculators.
>
> Anonymous, 1985

3. Is there a particular assessment we should use?

We ask our students to write the answer to these questions:

When do you use multiplication?
When do you use division?

If our students cannot write, we interview them instead.

The answers that our students give us tell us if they have learned that multiplication and division have more uses than simply finding answers to problems on a workbook page.