Chapter 11

Fractions, Ratios, Money, Decimals and Percent

Table of Contents

Before We Begin
Fractions....240
Guides and nothing more....240
Lesson One
Playing with the blocks....241
Fraction lesson, too....242
A rule that we present....242
A different one....242
Know the words to say....243
Proving blocks....243
Lesson Two
Two and one-half....244
Three and one-half....245
Beginning Number, Lesson Eight....245
Right triangles....245
Not right....246
Four triangles....246
Our students learn....246
Lesson Three
People fractions....247
Easy to say....247
Adding people fractions....247
Our assessment....248
Subtracting-people fractions....248
Let people help....249
Lesson Four
Fifty-four marbles....249
Dividing, cutting, sharing....250
Fractions everywhere....251
The awareness we create....251
The assessment....252
Lesson Five
Painting the house....252
Equivalencies....253
One is what we say it is....253
See it once, see it again....255
Lesson Six
Cube sticks....256
Patterns waiting to be seen....257
What happens, depends....258
Factors....259
Problems....260
What's the point?....261
Lesson Seven
Measurements....261
Variations....262
Assessment....263
Lesson Eight
Ratios....263
Science or mathematics....264
Shadow ratios....264
Bouncing balls....265
Diagonals of rectangles....265
Jumping farthest....265
Other opportunities....265
Lesson Nine
Rates....265
A single tank of gas....266
How long to reach Los Angeles?....267

Rate tables 267
Lesson Ten
Focus on an understanding 268
One is what we say it is, again 268
Where the decimal goes 269
New equivalencies 270
A rule for multiplying...270
What the answers mean 271
Ten by ten 272
Lesson Eleven
Not equivalent 273
The difference 274
3
Calculator key 274
Knowledge we can refine..274
Lesson Twelve
Fast finishers 276
Vending machine..277
Do we know to teach the parents? 277
Milk money 278
Restaurant lunch 278
Stores 279
Think money 279
Assessments 280
Summary
Twelve lessons 280
Questions from Teachers
1. How do we assess for fractions? What kind of assessments might we use? 280
2. In Lesson Ten, for the multiplication problems on the geoboard, the fractions in the answers were not reduced. Why not? Isn't the teaching of the reducing of fractions an extension of equivalencies?...281

Before We Begin

Fractions...

Fractions are:

Areas of shapes.	Measurements of time.
Parts of groups.	Partial lengths, weights, or heights.
Money's change.	Costs per unit sold.
Ratios of numbers.	Averages.
Decimals.	Percents.
Fragments.	Portions.
Equal shares.	Numbers waiting to divide.

Parent speaking to two children about to share a single piece of cake: One of you can cut the piece of cake; the other one can choose his piece first.

Children are familiar with fractions long before they come to school. How carefully will the cutter cut the cake? Do the children dividing the cake know the piece each one receives is called a half? Is there any question that the child doing the cutting and the child choosing first understand the meaning of dividing evenly?

Fractions are a part of children's lives before children even know that fractions are a part of math. Our hope is that the lessons that we teach leave our students as comfortable with fractions as our students were before our lessons began.

Guides and nothing more...

"Lift the Bowl" is a game in *Mathematics Their Way* (pages 181-183) in which the first player selects a beginning number of blocks. This first player then hides some of the blocks beneath the bowl and puts the rest on top. The second player, when told the beginning number, looks at the blocks on top and says how many blocks are hidden. For the illustration below, five blocks is the starting number.

(illustration 11-0-1)
(A bowl from the bowl game with three blocks on top.)

Playing the game gives young children the opportunity to think about combinations of two numbers that add to make another. In this example, the student learns "Three and two make five."

A visitor unfamiliar with "Lift the Bowl" was visiting a room where children were playing the game. The first two children she observed were playing it a different way. The starting number of blocks was five, but one child had placed a block on the top of the bowl and held two more blocks in his hand.

First student: There is a block on top and two more in my hand. How many blocks are hiding underneath?
Second student: Two.

When the visitor had finished her walk around the room she asked the teacher to tell her the name of the wonderful game the first two children had been playing. The teacher said, "Describe the game, and I'll tell you what it's called." When the visitor described the game, the teacher said, "Oh no! If I had seen the children playing it that way, I would have stopped them at once. That isn't what *Math Their Way* says to do at all!"

Can a book anticipate every learning experience we might face? Can a book tell us what is math and what is not? Books should no more prescribe a path of learning from which we must not deviate than we should prevent students from going beyond what we have told them to do. Can we teach thinking if no thinking is allowed?

The lessons that follow are guides for teaching about fractions and their decimal cousins. They are guides and nothing more.

Lesson One

Purpose	Learn about fractions informally. Learn the words to say and the numbers to write.
Summary	Students explore Power Blocks, guided by the questions that we ask. We teach words and numbers that describe the fractions.
Materials	Power Blocks, recording paper, blackline matrix.
Topic	Freely exploring, with learning directed through questions asked.
Topic	Which pieces make into other pieces?
Topic	S-1 = 1, T-1 = 1.
Topic	S-5 = 1.
Topic	Matrix for everything = 1.
Topic	Learning the words to say for fractions.

Playing with the blocks...

Principal to kindergarten teacher: Try these Power Blocks with the children in your room and see how you like them.

Two weeks later, the kindergarten teacher returns the blocks to her principal.

Kindergarten teacher to principal: These blocks must be meant for much older children. My kids could not find anything to do with them but play.

What do children learn by playing with the blocks? As they free explore, they create learning for themselves. They form patterns and make connections. They develop beginning number skills. They sort and classify and expand the language that they use. They form concepts of shapes and their relationships to one another as they construct geometrically. They measure as they build.

Our students' first fraction lesson is playing with the blocks. We channel their learning by the questions that we ask:

How many different pieces are in your set?
What does the word *different* mean? Different size? Different shape?
Do the codes written on the pieces give us any clue?
What meanings do the codes have for us?
Is there a pattern to the numbers and the letters?
Which of the same pieces make into larger sames? Which do not?

(illustration 11-1-1)
(Show what is meant by "larger sames".)

Which two of any piece make into a bigger piece within the set? Which three? Which four?

(illustration 11-1-2)
(Show examples of Power Blocks used to make bigger pieces in the set.)

How many different ways to make a piece? Trace the ways you find.

(illustration 11-1-3)
(Show what is meant by "how many ways" so that it is clear that for this question the pieces do not all have to be the same size. Show examples in blocks and traced.)

If the S-1 square has an area of one, how many other shapes (or pieces) can you find an area for? Keep track of all the areas that you find. Keep a record of your proofs.

(illustration 11-1-4)
(Show the S-1 square and other shapes with the proofs of their areas given.)

(illustration 11-1-5)
(As a separate illustration, show the indirect proof necessary to prove the area of the S-2 shape and then the S-4. Comment that these pieces and the long thin triangles and rectangles are in the set for just this kind of proof.)

What happens if we now say the T-1 piece has an area of one? Keep track of all the areas that you find. Keep a record of your proofs.

(illustration 11-1-6)
(Show the T-1 triangle and other shapes with the proofs of their areas given.)

Fraction lesson, too...

What do children learn by playing with the blocks? They learn about fractions as naturally as they learn about fractions outside of school.

Teacher: Today, we will say the S-5 square has an area of one. What is the area of all the other pieces now? Keep track of all the areas that you find. Keep a record of your proofs.

(illustration 11-1-7)
(Show the recordings done three ways: by tracing, by writing the formulas underneath... i.e. T - 5 + T - 5 = S - 5 and by writing out the fractional equivalents.)

A rule that we present...

(illustration 11-1-8)
(Show three blocks formed in the same shape as the S-5 square. The three pieces are a half (R-4) and two fourths (two S-3's).)

The largest square is one. Does that make each of the three pieces above equal to a third? We know that it does not. A child dividing something with a friend may comment that the other person got the bigger half. Bigger halves are not a part of the fractions that we teach. The fractions that equal the square in the illustration above are 1/4 + 1/4 + 1/2, not 1/3 + 1/3 + 1/3. A rule to help our students understand fractions is that, for fractions to have the same value, every part must be of equal size.

(illustration 11-1-9)
(S-5 made of two R-4's. S-5 made of four S-3's. Label the R-4's as halves. Label the S-3's as fourths. S-5 made of one R-4 and two S-3's, with each fractional part correctly labeled.)

A different one...

Teacher: What happens if we pick a different piece to have an area of one? How many different sizes can we say has an area of one?

(illustration 11-1-10)
(Show a matrix of the value of the different pieces as the value of "one" is changed.)

Teacher: If this piece is one-fourth the area of one, which piece would have an area of one?

(illustration 11-1-11)
(Show the T-3 triangle. Print the T-3 label next to it.)

Teacher: Can more than one piece have an area of one when T-3 is one-fourth?

For each question we ask, we encourage our students to reason out answers and to share their reasoning with everyone in class.

Know the words to say...

(illustration 11-1-12)
(S-5 square with two T-5 triangles next to it formed into a square the same shape.)

Teacher: How many equal parts make up this whole?
Students: Two.
Teacher: Does anyone know what we call each of the equal parts?
Student: A half.

If no one knows, we teach the name to use.

Teacher: There are two parts. We write the fraction for one part like this:

$$\frac{1}{2}$$

The bottom number is how many equal pieces together make up the whole. The top number is the number of pieces that we have.

(illustration 11-1-13)
(S-5 square with four S-3 squares next to it formed into a square the same shape.)

Teacher: How many equal parts make up this whole?
Students: Four.
Teacher: Does anyone know what we call each of the equal parts?
Student: Fourths.

If no one knows, we teach the name to use.

Teacher: Here's the fraction for a fourth.

$$\frac{1}{4}$$

What does the bottom number mean?
Student: How many pieces to make the whole.
Teacher: What does the top number mean?
Student: That's for the one little piece.
Teacher: Let's try another one.

We do not worry about reducing fractions to their lowest denominators. We leave 2/4 as 2/4 for now.

Proving blocks...

High school students learn about axioms, theorems and proofs in geometry. An axiom is a statement that is accepted as true without proof. A theorem is a statement that can be proved.

Axiom: The S-1 square has an area of one.

Theorem: Find the areas of the other shapes. Keep a record of your proofs.

Each proven area is a theorem that follows from the axiom.

New axiom: The T-1 triangle has an area of one.

When the axiom is changed, do the theorems change as well?

Power Blocks provide our students a background for understanding fractions regardless of the child's age. The beginning understandings of fractions come from the questions we ask and the experiences that the materials provide. There are no rules to memorize. There are no workbook pages to fill out.

Power Blocks are proving blocks. We define the problem. Our students seek out answers and prove the answers that they find. The lesson might be called fractions, geometry, or free exploring with the blocks. The lesson's name does not limit the learning being done.

Lesson Two

Purpose	Learn to find areas of shapes on geoboards. Learn to prove the areas of found shapes.
Summary	Students make shapes on their geoboards and learn specific techniques for proving areas. Areas of triangles are searched specifically for patterns.
Materials	Geoboards, geoboard recording paper; paper triangles and squares or Power Blocks for proofs.
Topic	Make shapes with areas of 2 1/2 and prove areas.
Topic	Make shapes with areas of 3 1/2 and prove areas.
Topic	Make shapes with areas of any size and prove areas.
Topic	Make right triangles and prove areas.
Topic	Find areas for and search for patterns in triangles with bases on the bottom row.
Homework	If geoboards can be sent home, finding areas of shapes is continued there.

Two and one-half...

Geoboards are an excellent material for teaching the geometry of shape. Geoboards are an excellent material for teaching fractional values of shapes, as well. But geoboards did not help Ryan, Hayley, or Jill make sense of shape or area (*Beginning Number*, Lesson Eight and *Geometry*, Lesson Two).

Power Blocks are a readiness material for geoboards. Power Blocks give children the background for understanding the shapes and the fractions of shapes they construct on geoboards. Power Blocks are Lesson One. Geoboards are Lesson Two.

Teacher: See if you can make shapes with areas of two and one-half square units on your geoboard. Use your square and triangle pieces of paper or use your Power Block pieces to prove the areas of the shapes that you create.

(illustration 11-2-1)
(Show a few different ways of making two and one-half square units. Put both the square and triangle paper and different size Power Block proofs along side each geoboard shape.)

In *Beginning Number*, Lesson Eight, (page 000), we asked our students to find ways to make shapes on their geoboards with areas of twos, threes and more. We find areas on geoboards by adding or subtracting wholes and halves. Our students added and subtracted fractions and whole numbers, even though we had not reached the point in this book where fractions were taught. Chapter titles and lesson numbers are not meant to limit what or when we teach.

Teacher: Please record the shapes you discover, so that you can keep images of the shapes you are inventing, even when you run out of room on your geoboard.

(illustration 11-2-2)
(Student recorded geoboard designs on paper. Label each shape "2 1/2".)

As our students work, we roam around the room asking individual children to prove the areas of their shapes to us. They also prove their shapes to one another.

Teacher: Prove the area each shape that you think has an area of two and one-half to your neighbor before you record it on your paper.
If your neighbor agrees with your proof, then your neighbor must sign his or her name next to your drawing.
If your neighbor does not accept your proof, then raise your hand and I will help you both decide if your proof is sufficient.

Students checking students means thirty teachers for every child in the room.

Three and one-half...

Teacher: Now, see if you can make shapes with areas of three and one-half square units on your geoboard. Please prove the area to your neighbor before recording it on your paper. Please also write the number of the area inside the shape.

(illustration 11-2-3)
(Student recorded geoboard designs on paper. Label each shape "3 1/2".)

***Beginning Number*, Lesson Eight...**

If we presented all of *Beginning Number*, Lesson Eight to our students, then we have already asked our students to find areas for any geoboard shape they might create. If we have not already asked, the time to ask is now. We may look back at Lesson Eight (page 000) to see examples of area-finding methods our students might employ.

Teacher: Work with a partner to see what kinds of shapes you can make on your geoboard for which you can prove the area. You may make shapes for any area you can prove.

Finding areas for any kind of shape is a fraction lesson automatically.

(illustration 11-2-4)
(Student recorded geoboard designs for a variety of shapes on paper. Label each shape with its proven area.)

As our students work, we walk around the room, sharing everyone's strategies for proofs.

Right triangles...

Teacher: Today I want you to see how many different right triangles you can find areas for. Right triangles are triangles with a right angle in them. (*Geometry*, Lesson Seven, page 000.) **The rule for the right triangles is that the base for each new triangle you make has to be on the bottom row of nails on your geoboard. Please record the triangles you make and the areas that you find.**

(illustration 11-2-5)
(All the right triangles are made with their bases on the bottom row of the geoboard. Give one example of a triangle made with its base not on the bottom row and add in the caption that this is not allowed. Demonstrate with the Power Blocks that right triangles are half of rectangles or squares. Then show on the geoboard right triangles as halves of rectangles or squares. Show a recording sheet with the areas written inside the triangles. All of the triangles on the recording sheet should have their bases on the bottom line of the recording area.)

Teacher: Let's see if we can find a pattern in the areas of the triangles you have made that might help us predict the areas for triangles not yet made.
For each area that you find, please write numbers for the height, the base and the area of the triangle on your chalkboard and hold it up for me to see. You already know how to find the number for the area. Find the number for the base and number for the height by counting the spaces and not the nails.

(illustration 11-2-6)
(Show geoboard examples of what is meant by *base* and *height* and *counting spaces*.)

Teacher: I'll record each new set of numbers on the overhead. Please look to see if I already have the numbers for your triangle written on the overhead before you write your numbers on your board.

Base	**Height**	**Area**
2	**4**	**4**
1	**3**	**1 1/2**
3	**1**	**1 1/2**
3	**3**	**4 1/2**
.	**.**	**.**

Will our students see the pattern? Will they see that area is found by multiplying the base times the height and dividing the resulting number in half? Can we be patient enough to wait until they do?

Not right...

Teacher: What is the height of these triangles?

(illustration 11-2-7)
(Show one acute triangle and one obtuse triangle. Show that, for the acute triangle, the height is the perpendicular line from the base to the highest point. Show that, for the obtuse triangle, the height is also the perpendicular line from the base to the highest point, but we need to extend the base beyond the triangle to find that perpendicular point. The height is how tall the triangle is, straight up from the line the base is on.)

Teacher: Find the areas for all the geoboard triangles that you can, even the ones that are not right triangles. Record their bases, heights and areas. Prove the areas that you find.
The rule is that the base for each triangle that you make has to be on the bottom row of nails.

(illustration 11-2-8)
(Several triangles on geoboards and recorded on paper. With their bases, heights and areas recorded in a table.)

The bases must be on the bottom row of the geoboards because measuring triangles made another way is more difficult. Which side of the triangle below shall we say is the base? What might be its height? What might be the measurements for each?

(illustration 11-2-9)
(Show a triangle on a geoboard with no sides parallel to the bottom row and no side lengths easily counted by nails touched.)

Mathematics is simple and basic and straightforward. The rules we add are designed to keep it so.

Teacher: Can you find a pattern in the areas of your triangles that might help you predict the areas for all triangles?
Is the pattern you find now anything like the pattern you found for right triangles?

Four triangles...

Teacher: Here are four triangles I have made.

(illustration 11-2-10)
(First triangle: Right triangle with a base of one and a height of four. The base extends from the first to the second nail of the bottom row. The height extends from the second nail of the bottom row to the second nail of the top row. The second triangle is exactly the same as the first except that the top of the triangle is now the third nail of the top row. The third triangle is exactly the same as the second except that the top of the triangle is now the fourth nail of the top row. The fourth triangle is exactly the same as the first except that the top of the triangle is now the fifth nail of the top row.)

Teacher: What is the base of each of these triangles?
Students: One.
Teacher: What is the height of each of these triangles?
Students: Four.
Teacher: What are the areas of each of these triangles? Work with a partner to find out.

Is there a pattern to be seen?

Our students learn...

When we ask our students to prove the areas of shapes on their geoboards, our students learn to use their problem-solving skills. They learn to find areas and prove the areas that they find. They learn from one another a variety of approaches to discovering new solutions. They learn from us techniques than might not occur to them. They learn that there are patterns in the answers to our questions. They also learn that using fractions at school is as easy and as natural as cutting cake at home.

Lesson Three

Purpose	Learn that fractions are special numbers describing part/whole relationships. Learn to add and subtract simple fractions.

Summary	Students learn to use people in the room to create simple fractions, then addition and subtraction problems. They also learn to create stories to accompany fractional numbers.
Materials	Students in the room, chalkboards, paper.
Topic	Fractions are created with people in the class.
Topic	Students create their own addition problems.
Topic	Students create their own subtraction problems.
Homework	What problems can our students create with the people in their home?

People fractions…

Teacher: How many people are in our class today? Count all the students in the room. Count me in as well.
Students: Thirty.
Teacher: What fraction of the people in the room are each of you?

The *thirty* in the answer given by the students represents the thirty people in the room, or the whole. But fractional numbers represent two different concepts at once—the whole and the part.

Teacher: Altogether, we are one whole class. The fraction of the class we are is the part we are of the whole. Thirty people are in the room. I am one of them. So, I am one-thirtieth of the people in the room. One out of thirty, or one-thirtieth, is written one over thirty, like this:

$$\frac{1}{30}$$

What fraction of the people in the room are you?

When the students know what fraction, or part, of the class they are, the teacher changes what is called the whole and asks again.

Teacher: Now, your row is one whole row.
How many people are there in your row?
What fraction of your row are you?
Use your chalkboards to show me how you think you write the fraction that you are.

Easy to say…

Eventually, our students will have to know the formal names for every fraction that they say or write: one over seven is one-seventh; one over two is one-half. But eventually does not mean right now. We can start our students out with fractions that are easy to say and easy to write.

Parts and wholes are not difficult to write. The number on the bottom (the denominator) is the number of the whole. The number on the top (the numerator) is the number for the parts. If we read the fraction as parts over wholes, parts and wholes are not difficult to say. One-eighth, one out of eight, or one over eight—if our students understand the meaning of the fraction, we accept anyway they say it at the start.

Adding people fractions…

Teacher: I am going to make up an addition problem for fractions. Cindy, how many people are in your row?
Student: Seven.
Teacher: What fraction of your row are you?
Student: One out of seven.
Teacher: How would you write that?
Student: One over seven.
Teacher: Show me on your chalkboard, please.
Student: 1/7
Teacher: What fraction of your row is Nicole?
Student: She's one over seven, too.
Teacher: What fraction of your row are you and Nicole together?
Student: Two over seven.
Teacher: We would write the addition problem like this:

$$\frac{1}{7} + \frac{1}{7} = \frac{2}{7}$$

Teacher: Now, everyone write this next problem that I make up on your individual chalkboards.
What fraction of his row is Daniel? Don't tell me. Write the fraction for Daniel on your boards.
What fraction of Daniel's row is Michael? Write that on your boards next to the Daniel fraction already there, with a plus sign between.
Now, what fraction of Daniel's row are Daniel and Michael together? Write that on your boards. Don't forget to put the equals sign where you think it goes. Hold up your boards so I can see what you have written.

The students record fraction problems on their individual chalkboards until the teacher feels they understand the process well enough to begin creating fraction problems on their own.

Teacher: Now, I want you to make up your own fraction problems for people. You may work together or by yourself. You may write about the people in the room or people from any other place you wish.
For each problem you create, please say what the whole group is before you write the fractions. Then either write a story or draw a picture to go along with the problems you create.

Our assessment...

$$\frac{2}{8} + \frac{3}{8} = \frac{5}{8}$$

Teacher: Who can give me a story to go along with these numbers?

When we tell our students to write the numbers for our problems, our assessment of our students' understanding is in the numbers that they write. When we ask our students to create problems on their own, our assessment is in the words and pictures that they write or draw. Our assessment of our students' understanding of the numbers that we write is in the stories they create.

As we do for all number operations, we connect concepts to symbols. Concept-connecting-symbolic means we use symbols to record events. Symbolic-connecting-concept, means we start with the symbols and construct what the symbols represent. Concept-connecting-symbolic, symbolic-connecting-concept—two-way street.

Teacher: Why is the answer to the 2/8 + 3/8 problem 5/8 and not 5/16? Why don't we add the bottom numbers to each other just like we add the top?

Can our students explain why we add numerators and not denominators? Is it possible for students to discover the rules for fractions for themselves?

Teacher: If a new child were to come to our class today, could you explain to him or her the rules for adding fractions? What do you think you might say?

Can our students verbalize or write out the knowledge that they have?

Subtracting-people fractions...
Teacher: How many people are in Brenda's row?
Students: Six.
Teacher: Brenda, Anthony and Russell, please stand up.
Teacher: What fraction of Brenda's row is standing up?
Students: Three out of six.
Teacher: Show me on your chalkboards, please.
Students: 3/6
Teacher: I'll write that on the overhead.
Three-sixths or three over six is the starting number in my subtraction problem. Anthony and Russell, please sit down.
What fraction of the row did I have sit down? Please write it on your chalkboards next to the three-sixths. Show me what you have.
Students: 3/6 2/6
Teacher: Put a minus sign in front of the two-sixths, so we can tell it is the number to be taken away. Show me what you have on your boards now.
Students: 3/6 - 2/6
Teacher: I'll write that on the overhead, as well.
What is the answer to this subtraction problem? What fraction of Brenda's row is still left standing after Russell and Anthony sat down? Write the fraction on your boards. Don't forget the equals sign.

Students: 3/6 - 2/6 = 1/6

The students record the fraction problems for subtraction on their individual chalkboards until the teacher feels they understand the process well enough to begin creating problems for themselves.

Teacher: Now, I want you to make up your own fraction problems for people. You may work together or by yourself. You may write about the people in the room or people from any other place you wish. You may make up subtraction problems or addition problems or problems that have addition and subtraction mixed in.

Please say what the whole group is before you write the fractions. You may either write a story or draw a picture to go along with each problem you create.

Let people help...

Fractions are special numbers that represent two concepts at once—the whole and the part. Students who are old enough to know that they are individuals at the same time that they are part of a whole group, are old enough to understand what people fractions mean.

Unless we are alone, we are in a group. We can ask people-fraction questions standing in the lunch line, waiting for the school bus, or sitting in a row of seats impatient for the assembly to begin. People-fraction questions can go home. What fraction of each family is each child? What fraction of each family is girls? What fraction is boys? What other people-fraction questions can our students ask at home? People fractions follow us wherever we go.

Learning about fractions is not something we save for an upper grade. We use Power Blocks and geoboards to help us make fractions a natural part of our students' learning from their earliest years in school. We let people help us, too.

Lesson Four

Purpose	Learn to be aware of fractions in life.
Summary	We ask our students to think about the sharing, cutting and dividing fractions in their lives.
Materials	None.
Topic	Teacher lead discussion on the sharing, cutting and dividing that lead to fractions in our lives.
Homework	Students bring examples from home to school to share.

Fifty-four marbles...

54 ÷ 4 is a division problem. The answer that a calculator gives is 13.5. But mathematics is more than numbers on a page or the answer that a calculator finds.

Teacher: Four children have 54 marbles to share among themselves. How many marbles will each child receive? Please be prepared to explain your answer to the class.

Patrick: The answer is 14 because when I do it on my calculator, I get 13.5, but we are supposed to round up when we get a .5, so its 14.

Calculators give us answers for the arithmetic that we do, but are calculators any use in giving students an awareness of what the numbers mean? Patrick has learned to round up when he sees .5, but has he learned to think about what the numbers mean?

Teacher: Patrick, what does your calculator say fourteen times four is?

Patrick: 56.

Teacher: Then if I said to you that four children each have fourteen marbles, how many marbles would they have altogether?

Patrick: 56.

Teacher: But the first problem I gave you said the four children have only 54 marbles, not 56. You would need to have started with 56 to give each child fourteen.

Rounding is a useful skill to know. Knowing when to round and when to not is another useful skill.

Jesse: You don't round. The answer is just 13.5. Everybody gets 13 and a half.

Teacher: How do you propose to give anyone half a marble?

Jesse: Hit the extra marbles with a hammer?

Teacher: Is that really how you would share any extra marbles that you had?

Jesse: No.

Aaron and Danielle: Each child gets thirteen marbles. Then they play rock-scissors-paper to see who gets the two left over. So three children get thirteen marbles and one gets fifteen.
Teacher: That way seems like it would work. Who has another way that the children might share?

The teacher questions each student's explanation to see if the students understand the numbers. Then the teacher uses the numbers 54 ÷ 4 to ask a different question.

Teacher: Four children have 54 cookies to share among themselves. How many cookies will each child receive? Please be prepared to explain your answer.

What is the difference between fifty-four cookies and fifty-four marbles? Individual marbles cannot be broken up and shared. Individual cookies can. Mathematics is more than numbers on a page.

Dividing, cutting, sharing...

Teacher: What are examples of sharing in your life?

Sharing food and drinks.
Sharing toys.
Sharing colored pencils or crayons.
Sharing math materials in class.
Sharing clothes.
Sharing the swing at recess.
Sharing secrets.
Sharing money.
Sharing comic books.
Sharing combs and brushes.
Sharing lipstick.
Sharing rides.
Sharing answers for homework.
Sharing answers secretly for a test.
Sharing ideas.
Sharing prizes won.
Sharing with the class.

Teacher: We know that candy bars can be shared fractionally. Can comic books? Can pencils? Which things do we divide using fractions and which do we divide another way? Can you explain your answer?

Teacher: What are examples of things you can cut?

Cutting cakes and pies.
Cutting firewood.
Cutting class.
Cutting up in class.
Cutting glass.
Cutting grass or hay.
Cutting someone down.
Cutting hair.
Cutting fingernails.
Cutting out paper dolls.
Cutting out the pattern for a shirt or dress.
Cutting with a knife.
Cutting down a tree.
Cutting back on spending.
Cutting through the water.
Cutting pictures from a magazine.
Cutting folded paper into snowflake designs.
Cutting diamonds.
Cutting cattle with a horse.

Teacher: We know that cut up cakes and pies can be described fractionally. Can cutting hair? Which of the things that we can cut can be described with fractions and which cannot? Explain your answers.

Teacher: What are examples of when you might use your skills of division?

Dividing the arithmetic problems on the workbook page.
Dividing to see if the multiplication answer was right.
Dividing handsful of squares into groups.
Dividing portions in the cafeteria.
Dividing into teams.
Dividing to find averages.
Dividing up the chores.
Dividing up the Sunday paper.
Dividing miles into gallons to see the mileage for the car.
Dividing up the land.
Dividing up the loot.
Dividing up the time.
Dividing up the day.
Dividing to find the cost of one.
Dividing to find your share.

Teacher: We know that we can divide squares into groups and if the groups don't divide evenly, the remainder is a fraction. Would dividing people into teams ever leave a fraction?
Which of the things that we can divide might have fractions in their answers at least sometimes? What would the fractions be? Which things never have fractions in their answers?
Explain your answers.

Fractions everywhere...

We ask our students to think about fractions in school. How do we help our students to think about fractions at home?

Teacher: Your homework assignment for tonight is to find examples of fractions and bring them with you tomorrow when you come to school.
You may bring in your examples in any way you can. Cut them out (ask permission first!), write them up, draw them, or bring them in your head.
Where do you think you might look to find examples? Where are fractions used?

If our students do not know where to look, we provide some clues. Examples come from the sharing, cutting and dividing discussions we have had in class. Where else might our students look at home? In newspapers, magazines and cookbooks. On packages for foods and drinks. Looking means listening, too. What fractions are used on the radio, on TV, or in conversations with relatives?

Examples can come from places outside the home. Store advertising signs, sizes for shoes and clothes, product names, street addresses, distances to freeway exits. Every situation that our students find can be the inspiration for a hundred situations more.

The awareness we create...

The object of this lesson is not the answers that we find, it is the awareness we create. Fractions are everywhere around.

We have three half-gallon bottles of soda to share for our party this afternoon. We also have paper cups for everyone. Can we figure out in advance if three bottles hold enough soda to give everybody in class at least one full cup? If any soda were left, how much would each child get for seconds? If three bottles are not enough, how many more will we need?
How should we cut Aaron and Kyle's birthday cake so that everyone gets a piece? What fraction of the cake will each piece be? Should we cut it differently, so that there will be cake left over for seconds? What fraction would these pieces be?
How shall we divide the class to have four teams for P. E. today? Will the teams come out evenly? If four teams does not give each team the same number of players and we want equal teams, how many teams should we have?
Julie found the fraction 1/3 in the newspaper for a one-third off sale. How could we tell the price of something that is now one-third less than its original price?

We ask questions for awareness. We create problems that are real. We take the risk of not knowing where the lesson may be going and let the lesson take us where it leads. Some problems may lead to work with fractions. Some may not. All involve thinking about how to use mathematics. All involve connecting math to life.

> Teachers ask questions for different reasons in the United States and in Japan. In the United States, the purpose of the question is to get an answer. In Japan, teachers pose questions to stimulate thought. A Japanese teacher considers a question to be a poor one if it elicits an immediate answer, for this indicates that students were not challenged to think.
>
> J. W. Stigler & H. W. Stevenson, How Asian Teachers Polish Each Lesson to Perfection, *American Educator*, Volume 15, Number 1, Spring 1991.

Is there a particular method for calculating the answers that we should teach our students? Can our students figure out if three bottles of soda are enough to share? Can they decide how to cut a cake so that everybody gets a piece? Can the children in our room divide themselves into teams? Can they show us what it means to take one-third off?

Answers are not the goal. Thinking is. Fractions are something that we think about.

The assessment...

If we do not have soda bottles, or cakes, our students show us how they would divide, cut, or share with manipulatives. Unifix Cubes, Power Blocks and water from the fountain are available for our students to act out how they might divide, cut, or share. The assessment for each problem that our students do is in the proofs they offer for the answers that they find.

What is the assessment for a lesson that may lead us in directions that we did not plan?

Teacher: Write down everything you have learned about fractions. If drawings help you show what you have learned, add drawings to the writing that you do. You may use your spelling notebooks if you are not sure how to spell a word.

Assessments do not have to involve special problems that we create. Assessments can be as simple as asking our students to tell us what they think they know. As we read their statements, we can see what has been learned and what we should teach next.

Lesson Five

Purpose	Learn about equivalencies.
Summary	Students use paper folding and Power Blocks to generate lists of equivalencies, which they then search for patterns.
Materials	Paper, Power Blocks, chalkboards.
Topic	Folding paper, recording the equivalencies formed.
Topic	Finding and recording Power Block equivalencies.
Homework	Paper folding can be shared at home.

Painting the house...

Three parents chatting in the stands at their children's hockey game:

First parent: The older my kids get, the harder it is for me to help them with their homework. Did you ever figure out the house-painting problem they brought home last night?
Second parent: I hate problems like that. I tried to think of all the algebra I could remember, but I didn't have a clue which numbers to put in which equations.
Third parent: What was the problem?
Second parent: One man can paint a house in three hours. A second man takes five hours to paint the same house. How many hours will it take both men to paint the house together?
Third parent: Well, you know it will take both the men less than three hours.
First parent: Why do you know that?
Third parent: Because if the first man can paint the house in three hours, it isn't going to take him as long to paint it with the other guy's help.
Second parent: How would you set the problem up to solve it in algebra?
Third parent: Well, how long does it take the first man to paint the house?
Second parent: Three hours.
Third parent. If it takes him three hours to paint the whole house, what fraction of the house would the man have painted in an hour?
First parent: A third?
Third parent: Right. He paints a third of the house in an hour. How much of the house does the second man paint in an hour if he takes five hours to paint the whole house?
Second parent: A fifth?

Third parent: So, the first man paints a third of the house in an hour, and the second man paints a fifth of the house in an hour. Working together, how much of the house would the two of them paint in just one hour?
First parent: A third plus a fifth?
Third parent: What's a third plus a fifth?
Second parent: Okay, we can't add a third to a fifth. We have to change both fractions to fifteenths. A third is 5/15 and a fifth is 3/15.
First parent: Why fifteenths? You lost me.
Third parent: We need a common denominator before we can add a third to a fifth. Fifteenths is the first fraction that a fifth and a third are equivalent to. What's 3/15 plus 5/15?
First parent: Eight fifteenths. But how do you know that it is 3/15 and 5/15?
Third parent: Oh, that's just the pattern for equivalencies. You multiply the two denominators together to get a common denominator. Then you multiply the numerators by the same number that you used to multiply the denominators. Since the numerators we started with were ones, the new numerators are just the numbers three and five, respectively.
First parent: Whatever.
Third parent. So, in the first hour they paint eight-fifteenths of the house. How much would they paint in two hours?
Second parent: Eight-fifteenth plus eight-fifteenths is sixteen-fifteenths.
Third parent. Sixteen-fifteenths is one fifteenth more than they need to do. Once they have painted the whole house, they stop.
Second parent: So it takes them an hour and seven-fifteenths to paint the house.
First parent: But where's the algebra?
Third parent: Why use algebra? All you need to know is fractions and equivalencies.

Equivalencies...

Teacher: What is the answer to this problem?

$$\frac{1}{2} + \frac{1}{4} =$$

Would our students know that the answer to this problem is 3/4 and not 2/6? The fractions in our students' lives come from sharing cookies, cutting cake, or dividing up marbles. Do our students ever use equivalencies?

If we meet in half an hour, is it the same if we meet in thirty minutes instead? Equivalencies are as common as the time of day. A half dollar and a quarter are 50¢ (or 50/100) and 25¢ (or 25/100), respectively. We use equivalencies to add the half and the quarter together to find their sum. Equivalencies are as common as the coins we carry in our purse or pocket. Equivalencies are a daily part of life.

One is what we say it is...

For the Power Blocks in Lesson One, our students learned that one is what we say it is. The S-1 square could as well be one as T-1 or S-5. For the people fractions of Lesson Three, our students also learned that one is what we say it is. One row, one class, one family.

Teacher: Take a single piece of paper and fold it carefully down the middle.

(illustration 11-5-1)
(Single piece of paper and the same piece of paper folded in half.)

Teacher: How many pieces of paper did I say for you to take?
Students: One.
Teacher: What do we call the pieces that we get from our careful fold?
Students: Halves.

If our students do not know the words we expect them to say, we teach the vocabulary.

Teacher: Please show me on your individual chalkboards how you would write a half.
Students: 1/2
Teacher: Fold your paper carefully in half again like this.

(illustration 11-5-2)
(Show the sheet folded into fourths.)

Teacher: Now, what do we call the pieces that we get from our second fold?
Students: Fourths.

We may teach our students to say fourths. We may also teach our students to say one over four.

Teacher: Please show me on your individual chalkboards how you would write a fourth.
Students: 1/4
Teacher: Fold your paper carefully in half again.

(illustration 11-5-3)
(Show the sheet folded into eighths.)

Teacher: What do we call the pieces that we get from this fold?
Students: Eighths.
Teacher: Please show me on your individual chalkboards how you would write an eighth.
Students: 1/8
Teacher: Fold your paper carefully in half again.

(illustration 11-5-4)
(Show the sheet folded into sixteenths.)

Teacher: Let's record what we have found so far. How many pages did we each start with?
Students: One.

1

Teacher: How many halves did we get from our first fold?
Students: Two.
Teacher: How would you write two halves?
Students: 2/2.

If our students do not know, we show them what two halves look like written out.

1 = 2/2

Teacher: How many fourths did we get from our second fold?
Students: Four.
Teacher: Show me how to write four fourths.
Students: 4/4.

1 = 2/2 = 4/4

The questioning continues. The writing continues, as well.

1 = 2/2 = 4/4 = 8/8 = 16/16

Teacher: Show me on your chalkboards how many fourths there are on one-half of your paper?
Students: 2/4

1/2 = 2/4

Teacher: How many eighths?
Students: 4/8

1/2 = 2/4 = 4/8

Teacher: What question am I going to ask next?
Student: How many sixteenths.

1/2 = 2/4 = 4/8 = 8/16

Teacher: How many eighths in a fourth?

1/4 = 2/8 = 4/16

Teacher: Look at the numerators and the denominators of all these fractions and see what patterns you can find. Share with the rest of us any patterns that you see.

Paper can be folded more ways than in half and half again.

Teacher: Let's try folding a piece of paper in a different way to see what we can see. The rule for folding is that each new fold we make must divide the paper equally. As we fold, we'll record the fractions that we find.

(illustration 11-5-5)
(Paper folded in thirds, then in half then in thirds again.)

1	**= 3/3**	**= 6/6**	**= 18/18**
1/3	**= 2/6**	**= 6/18**	
1/6	**= 3/18**		

Teacher: What happens to the size of the denominator as the number of folds increases? Are there patterns that you can see? Can you use the patterns to help you predict what the numbers will be before you fold the paper to find out?

Once the students understand paper folding, they fold new pieces and keep track of the fractions that their folds produce.

Some fractions have the same value as other fractions, some do not: 1/3 is equivalent to 2/6, but 1/3 is not equivalent to 1/2. What makes some fractions equivalent to others and some not are rules to be discovered and patterns to be seen.

See it once, see it again...

Teacher: The S-5 square is one. Make S-5 with two pieces that are equal in size.

(illustration 11-5-6)
(S-5 with two T-5 blocks alongside made into the S-5 shape and two R-4 rectangles also made into the S-5 shape.)

Teacher: I'm going to keep a written record of what we find.

1 = 2/2

Leave the halves that you have found next to your S-5 shape. Is there any way to divide the S-5 square into thirds? Remember, all the pieces have to be the same size or the same area.
Students: (After trying for awhile) No.
Teacher: Is there any way to divide the S-5 square into fourths?
Students: Yes.
Teacher: Show me all the ways that you can find.

(illustration 11-5-7)
(Show the T-4, T-8, R-2 and S-3 shapes as fourths, with the halves still visible.)

Teacher: I'll add what you have found to my list.

1 = 2/2 = 4/4
1/2 = 2/4

What is the next fraction that we can divide the S-5 square into?
Students: (After trying for awhile) Eighths.
Teacher: Show me all the ways that you can find.

(illustration 11-5-8)
(Show the T-7, T-8 and R-1 shapes as eighths, with the halves and fourths still visible.)

Teacher: I'll add what you have found to my list.

1 = 2/2 = 4/4 = 8/8
1/2 = 2/4 = 4/8
1/4 = 2/8

Teacher: Look at the numerators and the denominators of all these fractions and see what patterns you can find. Share with the rest of us any patterns that you see.

Our students saw these same fraction patterns when they folded paper in half and then in half again.

Teacher: The next fraction that we can divide the S-5 square into is sixteenths. Can you use the patterns that you see to help you predict the numbers we will record before you use your blocks to find out?

In mathematics we look for patterns everywhere. Equivalence is a pattern we can learn to see.

Lesson Six

Purpose	Learn to use equivalencies to find factors.
Summary	Students use Unifix Cube sticks to generate lists of equivalencies which, combined with Start with, go bys, they search for patterns to help them make sense of factors.
Materials	Unifix Cubes, butcher paper, Start with, go by chart, Power Blocks.
Topic	Unifix equivalencies are recorded and explored.
Topic	Finding common denominators to use to add unlike fractions. Specific techniques taught depend on the patterns that the students see and the sense they make out of their equivalency experiences.

Cube sticks...

Folded paper and Power Blocks give our students an opportunity to look for patterns in equivalencies. Unifix Cube sticks expand the opportunity.

Teacher: Make one cube stick eight cubes long.

(illustration 11-6-1)
(Unifix Cube stick eight cubes long.)

Teacher: How many sticks did I ask you to make?
Students: One.
Teacher: One is what we say it is. This time one is the cube stick that you just made. How many different fractions can you break your cube stick into? Remember, the rule for fractions is that every part must be of equal size. For your cube sticks, this means the way you break your cube stick up must give you little stacks of cubes all of equal size.
We'll do this first cube stick together, so I can show you what I mean.

(illustration 11-6-2)
(Show the ways to break the cube stick into fractions, with each way labeled. An eight stick breaks into halves and fourths and eighths.)

Teacher: I'll write the fractions we have found.

1/2
1/4
1/8

Everyone make a cube stick nine cubes long. One is now a cube stick nine cubes long. How many different fractions can you break this stick into? Remember, every part must be of equal size.
Show me all the ways that each of you can find.

(illustration 11-6-3)
(Unifix Cube stick nine cubes long. Show the ways to break the cube stick into fractions, with each way labeled. A nine stick breaks into thirds and ninths.)

1/3
1/9

Teacher: Now see what different fractions you can find for a cube stick ten cubes long.

(illustration 11-6-4)
(Unifix Cube stick ten cubes long. Show the ways to break the cube stick into fractions, with each way labeled.)

1/2
1/5
1/10

Teacher: Now see what you can find for a cube stick of eleven cubes.

1/11

Longer sticks do not always mean there are more fractions to be found. A cube stick of eleven cubes makes elevenths and nothing more.

(illustration 11-6-5)
(Spaces numbered from one to 36 marked off on a big piece of butcher paper for recording the fractions for the cube sticks. The numbers are at the top of columns marked off to define the writing space for the fractions to be written beneath each number. The fraction numbers for eight, nine, ten and eleven are already written on the paper beneath their respective 8, 9, 10 and 11. Include in the caption that as much space is marked off as is available. Any number from 30 to 60 is a good number with which to work. More than 60 is nice if there is space. Also note in the caption that we put this on butcher paper and not on the chalkboard so we may save it more easily.)

Teacher: Up until now, I have been doing all the recording. Now it is your turn to write the fractions for the sticks.
You and your workmate can decide with which cube-stick lengths you wish to work. For any cube-stick length you choose to be one, write the fractions that you find in the space beneath its number on the paper at the front of the room.
Before you write your numbers on the paper, see if someone else has already written fractions there. If they have, then see if you agree with what they have written, and write in only the new fractions you have found.
Write large enough so that everyone can see what you have written, but not so large that your numbers go into the next column.
When you find all the fractions that you can for a cube stick, select another cube stick and find fractions for it.
Are there any questions? Then you may begin.

We may think our instructions are clear. We may even think that the absence of questions from our students is a measure of the power of our words to communicate. We know, however, that the evidence of the clarity of our words is in the work our students do. They will show us soon enough if we need to use a different set of words to make our meaning clear.

Patterns waiting to be seen...

The numbers that our students write are patterns waiting to be seen.

(illustration 11-6-6)
(Fractions and equivalencies beneath the number on the butcher paper.)

What kinds of questions might we ask for the numbers on the paper?

Is there a pattern for which cube sticks have halves as fractions?
Is there a pattern for the thirds and fourths and fifths?
Is there a pattern to the pattern?
Can you use the pattern to see any fractions that we might have forgotten to include?
Can you use the pattern to tell what fractions might be made from cube sticks whose lengths are not written on the paper?
Which sticks have the most fractions?
Which have the fewest?
Why?

One pattern that our students are likely to see is that halves occur every second stick, thirds occur every third stick, fourths occur every fourth, fifths occur every fifth and so on. Another observation is that the sticks that break into the most fractions are not necessarily the longest sticks. The twelve-cube stick breaks into many more fractions than does the thirteen-cube stick. The number of fractions that can be made is not a function of a stick's length.

What happens, depends...

1/2 + 1/4 =

Teacher: Can you add one-half to one-fourth? If you can, can you prove the answer that you find?

What happens now depends on what has happened before. What have our students understood of equivalencies? What patterns have they seen? Can they put their understanding to use without being told specifically what to do? Can they discover for themselves the patterns for adding fractions with denominators that are not the same? Will the patterns they discover serve them as well for the next problem that we pose?

Teacher: What is 1/6 plus 1/7?

Our students may have seen patterns in their cube sticks that will lead them collectively to see how they might add one-sixth to one-seventh. However, if our students need assistance in making the connections, we start with 1/2 + 1/4.

Teacher: Let's see if we can figure out what cube stick to use to add one-half to one-fourth. What is the shortest stick that will break into halves and fourths?

Is the chart for fractions that our students made still posted on the wall? If not, there is another way to calculate the shortest stick.

Teacher: Let's use Start with, go by, both to find the stick to use. (*Beginning Addition and Subtraction*, Lesson Two, page 000.) **Take the denominators for each fraction as the Start with, go by numbers and see which number comes up first in both columns.**

2	**4**
4	8
6	12
8	16
10	20
12	24

Teacher: Which number is the first to appear in both columns?
Students: Four.
Teacher: Please make two cube sticks four cubes long. Each of these cube sticks is *one*.
Show me one-half of the first stick and one-fourth of the second stick.
Now add the half to the fourth.

(illustration 11-6-7)
(Two cube sticks, four cubes long. Each stick is a different color. The first stick broken in half. The second stick broken into fourths. One of the halves and one of the fourths separated out, with the half added to the fourth.)

Teacher: What is one-half plus one-fourth?
Students: Three-fourths.

Our students may understand that the answer is three-fourths, or we may have to teach them that each cube for each cube stick in our example is a fourth. Three cubes snapped together make three-fourths. What happens depends on what our students need to know and on what they understand.

1/4 + 2/3 =

Teacher: Now, let's see if we can figure out what cube stick to use to add one-fourth to two-thirds. How do we find the shortest stick that breaks into fourths and thirds?

(illustration 11-6-8)

(Show all of the steps of the 1/4 + 2/3 problem worked out with cubes and recorded in numbers. Use cube sticks of two different colors. Also show the start with, go bys.)

Whether our students know that twelve is the common multiple for thirds and fourths or they use Start with, go bys to find out, depends on the patterns they have seen and the connections they have made. We watch to see if our students can solve problems on their own or if they need us to guide them through the steps.

What happens in any lesson always depends on what has gone before. Our students show us when they need our help and when they know enough to make connections for themselves.

Factors...

In *Beginning Addition and Subtraction*, Lesson Two, we asked our students what patterns they could see in the both numbers if they knew the two Start with, go by numbers. We saved their Start with, go by chart for a later time. That later time is now. The pattern is the same. If our students could not see the pattern then, we help them see it now.

Teacher: Factors are the numbers that we multiply together to get a bigger number. Six is the first number on our chart in both columns. What numbers multiply together to get six?
Student: Two times three.
Teacher: Okay. Any other numbers?
Student: One times six.
Teacher: Any other numbers?
Student: Three times two.
Teacher: True, but we'll count 3 x 2 as the same as 2 x 3 and 6 x 1 as the same as 1 x 6. The factors of six are 1, 2, 3 and 6.
Any other numbers?
Student: No.
Teacher: Four is the next both number. What are all the different numbers that multiply together to get four?
Student: Four times one and two times two.
Teacher: So the factors are 1, 2, and 4.
Let's record what we are finding out. I won't record the ones, since one is a factor for every whole number.

Start with	Start with	Both	Both factors
2	3	6	2,3,6
2	4	4	2,4
2	5	10	2,5,10
3	3	3	3
3	4	12	2,3,4,6,12
3	6	6	2,3,6
4	5	20	2,4,5,20
4	6	12	2,3,4,6,12
4	7	28	2,4,7,14,28
4	8	8	2,4,8

By adding factors to the chart, our students have more information to use in their pattern search.

Teacher: What patterns can you see in the column of both factors on this chart?
Student: All the numbers have themselves as a factor.
Student: Three has the fewest factors.
Student: Twelve and twenty-eight have the most.
Teacher: What patterns can you see that might help you know what the both numbers would be if you knew the two start-with numbers?

What was the answer that our students gave when we asked this same question in *Beginning Addition and Subtraction*, Lesson Two? What might be their answer now?

Student: The Start with, go by numbers are factors of the both numbers.
Teacher: Two and three are factors of six. But two and three are also factors of twelve. Why do you think six is the first both number for two and three and not twelve?
Student: Because you can divide two and three into six.
Teacher: But, you can divide two and three into twelve.

Student: But twelve is bigger. The both number is the smallest number both start-with numbers can divide into! Look! That's true for all the start-with numbers on the chart!
Teacher: That seems to be true for all the numbers on our chart so far. Let's find some more start-with and both numbers to see if the pattern holds up.

Factors are a pattern we can see in Start with, go bys or in dividing numbers or in cubes. Factors are also numbers to use for adding or subtracting fractions when the fractions do not have denominators that are the same. Can our students use the patterns they see now to help them know the denominators to use for equivalencies?

Problems...

When we teach a concept, how can we ensure the concept is understood? We use a range of problems to expose our students to the variety of possibilities that exist. Adding 1/4 to 1/2 is not the same as adding 1/4 to 1/3. Adding 1/6 to 1/7 is different, still. We use materials and situations in our class and outside of school to create problems that have meaning for the students in our room.

With Power Blocks and cubes:

(illustration 11-6-9)
(One Power Block S-5 square by itself. Next to it, an S-5 square made up of an assortment of pieces. List the specific pieces, so the illustration matches the numbers below.)

Teacher: The S-5 square is one. What are the fractional values of all the other pieces I have used to make the square?

1 = 1/2 + 1/4 + 1/8 + 2/16

The equation we have written is for Power Blocks. Can you prove to me that 1/2 + 1/4 + 1/8 + 2/16 = 1 using cubes?

With paper folding and cubes:

(illustration 11-6-10)
(A paper folded in half with 1/2 written on one half. The same paper folded in half again with 1/4 written in one of the fourths created by the new fold. The 1/4 is NOT written on any part of the section already labeled 1/2. The same paper folded into thirds (meaning the smallest section is now twelfths). 1/12 is written in each of the twelfth sections that were not previously labeled 1/2 or 1/4. The equation *1/2 + 1/4 + 3/12 = 1* is written beneath the paper foldings.)

Teacher: Can you prove that 1/2 + 1/4 + 3/12 = 1 with cubes?

From word problems that our students create:

Teacher: What is the answer to this problem?

1/4 + 1/2

Student: Three-fourths.
Teacher: Who would like to tell me a story to go with these numbers? Brenda.
Brenda: I ate a half a pizza, then I ate another fourth, so I ate three fourths of the pizza.
Teacher: How much of the pizza would be left?
Brenda: One-fourth.
Teacher: Let's draw a picture to see if Brenda's story matches the numbers on the board.

(illustration 11-6-11)
(Drawing of stick figure Brenda eating a half and then a fourth of a pizza.)

Teacher: Now, let's see what other kinds of adding and subtracting stories we can invent. You can use your imagination or you can see if you can think of problems that are real.

The students write and draw. The teacher adds words to the spelling notebooks as needed.

From questions we might ask:

Teacher: How many examples of things that are one-half of something can you think of?