Chapter 12

Advanced Addition and Subtraction

Table of Contents

Before We Begin

A hundred years from now...

Students in a local grade school were given the assignment of describing what kinds of technology the world would have 100 years away. They did their very best to imagine a world a century from now. But how good could their best guesses have been?

Let us imagine that this same assignment was given to grade-school students 100 years ago. These ancient prognosticators would be about 105 or 110 years old by now. How would these grade-school forecasters have done at anticipating the technologies in our world today? Would their teacher's guesses have been more accurate?

Let us take something that existed 100 years ago and see how well we think the old-time students might have imagined what a hundred years of change might bring. We know from history that cameras were in common use 100 years ago. Let's imagine what the century-ago children and their teacher might have predicted the future of the camera might bring. Would they have foretold the following events?

Electricity in batteries and bulbs for flashing light.
Color photos.
Moving pictures.
Talking pictures.
Color pictures that moved and talked.
Pictures that developed by themselves.
Talking, moving pictures sent through the air and received via television in every home.
Live satellite coverage on TV of events around the world.
Hand-held video recorders and VCRs.
Remote controls.

How far down this list can we speculate that the imaginations of the grade-school children and their teacher of a hundred years ago would have carried them?

Typewriters existed 100 years ago. What innovations would the first typewriters have led an imaginative child or teacher to predict?

Electric typewriters.
Carbon paper for making many copies.
Typewriter paper easy to erase.
Typewriters that erased the paper for us.
Xerox machines for making instant copies of what we type.
Computers with word processors and spelling checkers.
Laser printers to print out the computer screen's contents.
Fax machines to send typing anywhere around the world.
Electronic mail to send the typing from computer to computer via satellite.

How far down this list for typewriters can we speculate that the imaginations of the children and their teacher would have carried them?

Cameras and typewriters existed 100 years ago. Their existence might have led the forecasters to anticipate some of the changes that have occurred. What technology existed a hundred years ago that would have led anyone to predict that there would be:

Home videos.
Cellular phones and phone answering machines.
Hand-held calculators with solar-powered batteries.
Bar codings read by scanners on practically everything.
Personal computers.
Fax machines.
Disks of music or of moving pictures read through a laser beam of light.
Microwave ovens.
Routine air travel by jumbo jet.
Missions to the moon.
Satellite TV.
Organ transplants.
Artificial hearts.
Nuclear bombs.

We can see the predicting difficulties for the students and their teacher 100 years ago. The difficulties for predictions are compounded a hundred-fold for students now. Changes in our world today come at a much accelerated rate. In the age in which we live, knowledge doubles every four years. Twice as much information is available to a senior graduating from high school or college as was available in the world when the student's freshman year began.

Could modern students do as well in describing changes in technology even ten or twenty years from now as students might have done 100 years ago? Would we as teachers be any better at predicting what the future will be like?

Reading and writing and thinking...

We cannot predict the future, but we know what skills will survive regardless of what the future brings. The skills we know our students will need are the skills of reading, writing and thinking.

Computers with spell-checking word processors, built-in thesauruses and grammar guides makes more of us writers than ever would have dared write before. Computers already recognize speech and carry out spoken commands. Writing in the future may not mean typing out words as we do now. Computers may translate our spoken words into print and verify the spelling of any words we say.

The computer will take the drudgery out of the writing process. It will make all our writing legible and spell words for us as if we all could spell. But a computer is no more than a tool in helping us express our ideas. A computer cannot be our mind. We must still acquire the art of expressing our thoughts and freeing our imagination. Although the form it takes will change, writing is still a skill students will develop in school and use everywhere they go.

Reading in the future may not always mean looking at letters and thinking what words these letters represent. Computers will translate the letters and words into sounds for us. The definition of reading may shift in emphasis from understanding printed words to drawing meaning from the words, regardless of their form.

The computer will take the drudgery out of the reading process for those of us who do not read well. It will make all words readable even for the dyslexic child. There may no longer be anyone who is "illiterate." But the computer is no more than a tool in helping us understand the meaning of the words we hear. Though the form it takes will change, reading is still a comprehension skill students will develop in school and use everywhere they go.

The form that thinking takes will not change. Thinking is what our minds are meant to do. But if we cannot think creatively and productively for ourselves, we will not be able to adapt to the rapidly changing world. Thinking is a skill our students will develop in school and use everywhere they go.

Where is mathematics on this list of skills our students will need as adults ten or twenty years from now? Mathematics taught as something to be memorized has already been replaced by a machine. Mathematics taught as a way of thinking is a skill students use everywhere they go.

Everywhere except in school...

Where is arithmetic on this list of future skills? Except in school, calculators already do arithmetic for us. We watch as the clerk in the fast-food store presses the keys for our order of hamburgers and

French fries. The keys on the machine have the prices preprogrammed. The clerk need not even calculate our change. Do we even need the button-pushing clerk to take our order? We could press the buttons ourselves and make the clerk obsolete.

We watch the bar-code scanner at the grocery store display the price and name as each object in our cart passes through the laser's beam. The scanner of the future may read the entire cart at once and print a list of everything we bought. Will we even need the grocery checker then?

What tricks of arithmetic can pocket-size computers perform? Pen a list of numbers to add on the computer's notepad screen, draw a line beneath the column and the answer appears. Change any number in the column, the answer now reflects the change.

We solve arithmetic problems on our own nearly every day and night, but how much arithmetic do we use and what kind? When do we add or subtract or multiply or divide each day? Is the arithmetic we use in life reflected in how and what we teach? Do we ever use a calculator? Do we ever not? Are we preparing our students for the future or preparing them for the past? Do we fight against the changes in the world or ready our students for a world of change?

Excuses...

Ninety-eight percent of our students, rich or poor, already have access to calculators. Yet how much has teaching changed to reflect what the calculator can do? How many of the following excuses do we hear for keeping calculators out of school?

Students will not learn to understand arithmetic if they can use a calculator instead.
What will students do when they do not have a calculator close at hand? They will not always have a calculator with them everywhere they go.
Students should not have to use a calculator when they shop to know exactly how much they have spent and how much change is fair.
Calculators will not be allowed when students take the year-end tests.

Are the excuses justified?

Students will not learn to understand arithmetic if they can use a calculator instead.

Calculators cannot do our understanding for us. We must still acquire the art of comprehending the answers we calculate. Daniel subtracted 7 from 33 and said the answer was 36 (page 122). Daniel was right 95% of the time. He found his answers using rules he had memorized. It was not a calculator lesson that kept Daniel from understanding what the numbers meant.

When our students understand numbers, their understanding is not diminished by finding answers with a calculator's aid. With a calculator in hand, students can spend more time inventing problems to solve and creating ways to solve them, and less time memorizing all the rules. The calculator used properly gives our students more time to think.

What will our students do when they do not have a calculator close at hand?

How often do we write numbers on a piece of paper and do the computations that we learned in school? How often do we use a little estimation or a little finger counting to get the answer instead?

We use whatever method helps us get the answer. We take whatever time we need. There is hardly any place outside of school where inventing methods to find answers is not allowed. There is hardly any place outside of school where arithmetic is a test of speed. We can all come up with answers to the problems that we face.

Students should not have to use a calculator when they shop to know exactly how much they have spent and how much change is fair.

When we are shopping, do we search for paper and a pencil to make a list of all the prices in our shopping cart? Or, if we care to find an answer, do we settle for an approximation instead? Do we round all the 95¢ and 99¢ numbers to the nearest dollar to make our calculations easy? Do we match a 59¢ item with another item in the middle range and say a dollar for the two is close enough? When is the last time we calculated the exact change we should anticipate? Do we doubt the calculations of the cash register that tells us the change we should expect?

If our students once grown-up need mathematical precision in their jobs, they will have a calculator or computer close at hand. If they want to balance their checkbook at home, a calculator will be available as well. If our grown-up students are curious to know how much they have spent before the cash register does the adding for them, an approximation of the total will suffice for them as it does for us.

Calculators will not be allowed when students take the year-end tests.

Shall we teach our children solely for the tests and ignore the world all around us? Shall we deny our students the opportunity to learn mathematics in school just because not all tests have kept pace with change. Let our students take their calculators with them to the test as they will in life. If use of a calculator makes a test obsolete, let the test makers revise their tests so that they are relevant to this world.

How many hours at school and at home do students spend preparing for spelling tests, as students have been doing for a hundred years or more? Have all the hours spent memorizing words ever changed a poor speller to a good? Memorizing lists of words does not make spellers out of those of us who cannot spell. Memorizing does not teach us how to think. Have the test makers given any thought to how we really learn to spell?

Advanced...

Our students already know how to add and subtract. The activities in the *Beginning Addition and Subtraction* chapter have already strengthened our students' skills. Our students can access calculators for finding answers to any problem, large or small. What more is there for us to teach? What more is there for our students to learn?

In *Beginning Addition and Subtraction* our students learned how to take handfuls of squares to create their own problems. They learned that problems seemingly in isolation were really problems in the middle of a stream. They learned to create and solve their own people problems and write stories for numbers. They learned about reality and reasonableness. But every answer to every problem, large or small, they found either by counting or with a calculator—no regrouping was involved.

Understanding numbers is more than knowing how to count. There is more to finding answers than a calculator can reveal. Students can count to 100 and not know that the two different zeros in 100 mean two different things. One zero means we have no ones, the other means we have no tens. But how can there be no ones or tens when we know that one-hundred ones and ten tens are in 100? Advanced addition and subtraction gives our students a better understanding of the structure of the numbers that we use.

Bases...

All the numbers that we use each day are written in base ten. Why should we teach our students about numbers in bases they will not use? Why would we ever need to know about base two or three or five?

We were taught base ten in school. We were also taught bases three, four, five, seven, twelve, thirty, sixty, 365 and a few more besides. It takes:

3 feet to make a yard.
4 quarters to make a dollar.
5 pennies to make a nickel.
5 fingers to make a fist.
7 days to make a week.
12 inches to make a foot.
12 months to make a year.
24 hours to make a day.
28, 29, 30, or 31 days to make a month.
60 seconds to make a minute.
60 minutes to make an hour.
365 days or so to make a year.

We have not mentioned pints to quarts, quarts to gallons, or dollars to pesos, yen, or pounds. We have not mentioned the hundred other ways we use different bases in our lives.

Remember Mr. Smith? (Page 150)

> Mr. Smith leaves his home at seven o'clock in the morning to walk to the store nine miles away. After fifty minutes, Mr. Smith has walked three miles. What time will it be when he gets to the store?

Teacher: When did Mr. Smith reach the store?

Half the class of students: 8:50.
The other half of the class: 9:30.

Which half was right? And how could the half that is wrong be in such complete agreement?

The logic of each group of students was equally correct. If Mr. Smith walked 3 miles in 50 minutes, then he would walk 9 miles in 150 minutes. But the first group of students thought only in base ten and added 150 to the starting time.

7:00
<u>1:50</u>
8:50

The second group of students knew that the 150 was in minutes and minutes are base sixty, not base ten. In base sixty, 150 is 2 groups of 60 and 30 not yet grouped. 150 minutes in base sixty is written as 2:30.

7:00
<u>2:30</u>
9:30

We use different bases all the time.

12 yds, 2 ft, 5 in	**3 hrs, 20 min, 17 sec**	**5 gal, 1 qt, 1 pt**
<u>+ 7 yds, 1 ft, 6 in</u>	**<u>+5 hrs, 15 min, 27 sec</u>**	**<u>+7 gal, 1 qt, 2 pt</u>**

Adding feet and inches, finding differences in time, combining pints and quarts. We added and subtracted measurements and time when we were in school. Did we learn to call these calculations "doing bases"? Can we learn to call these problems "doing bases" now?

The computers that surround us make common use of bases two, eight, sixteen and thirty-two. Calculator and computer programs for engineers and programmers convert numbers between binary (base two), octal (base eight), decimal (base ten) and hexadecimal (base sixteen). Learning different bases in school makes our students comfortable with any of the bases they use in their lives outside of school.

Lesson One

Purpose	Learn to search for patterns in bases other than ten.
Summary	Student record and examine plus one and minus one patterns in different bases.
Materials	Trading boards, squares, cups, bowls, Blacklines for recording strips.
Topic	Squares and cups: base 4, base 5, base 6, if needed base 3. Then squares, cups and bowls as the base cycle is repeated. Then Base ten.
Topic	+ and - 1 with squares and cups, base 4.
Topic	+ and - 1 with squares and cups, base 5.
Topic	+ and - 1 with squares and cups, base 6.
Topic	+ and - 1 with squares and cups, base 3.
Topic	+ and - with squares, cups and bowls, base 4.
Topic	+ and - with squares, cups and bowls, base 5.
Topic	+ and - with squares, cups and bowls, base 6.
Topic	+ and - with squares, cups and bowls, base 3.
Topic	+ and - with squares, cups and bowls, base ten.
Homework	If students need more practice, cups and bowls can be sent home, but only with children who understand the basic process.

Plus and minus one…

Base means "What number are we grouping by?" Base ten means we are grouping by tens. Base three means we are grouping by threes. Using different bases is no more complex than knowing the groups to use.

Teacher: You will each need squares, cups and a piece of paper for this activity.
Draw a line down the middle of your paper like this.
On the top right half of your paper, draw a square. You may trace one of your squares if you wish. On the top left half, draw a cup. You may trace the cup, if you wish. Don't worry if you don't think you are great at drawing. All we need to be able to tell is which of your drawings is the cup and which is the square.

(illustration 12-1-1)
(Blank piece of paper with a line down the middle. Drawing of the square and the cup on the paper. Square on the right, cup on the left. The illustrations are meant to parallel the dialog. Dialog in one column, illustrations in an adjacent, parallel column.)

Teacher: We are going to play a base game called "plus one." Base means the number we are grouping by. Today we will be grouping by four, so the base is called four. You'll see what I mean in just a little bit.
For this game, whenever I say "plus one," except just then, please put one square on the right side of your trading board. We call this piece of paper the trading board.
Plus one.

The teacher tells and shows the students what to do.

(illustration 12-1-2)
(One Power Block S-1 square correctly placed on the right side of the trading board.)

Teacher: Have your neighbor check you and you check your neighbor so you can tell me if my instructions have been clear so far.

Telling, showing, checking. We use all the resources at our disposal to insure we leave no one behind.

Teacher: Read me the number of squares you have on your trading board.
Students: One.
Teacher: Plus one.

(illustration 12-1-3)
(Two squares correctly placed on a trading board.)

Teacher: Check your neighbor and have your neighbor check you. Read me the number of squares you have on your trading board.
Students: Two.
Teacher: Plus one.

(illustration 12-1-4)
(Three squares correctly placed on a trading board.)

Teacher: Check your neighbor and have your neighbor check you. Read me the number of squares you have on your trading board.
Students: Three.
Teacher: Plus one.

(illustration 12-1-5)
(Four squares correctly placed on a trading board.)

Teacher: What base did I say this was?
Student: Base four.
Teacher: The rule for bases is that we cannot have any of the base numbers lying around loose. Any time we get as many as the base number, we have to gather them up and put them in something. So, for base four, we gather up the four squares and put them in a cup, like this.

(illustration 12-1-6)
(Four squares placed in a cup on the left hand side of the trading board.)

Teacher: How many cups do you have on your trading board?
Students: One.
Teacher: We read that as one cup. How many squares do you have left on your trading board?
Student: We don't have any.
Teacher: We read that as zero. Please read me what you have on your trading board.

Students: Zero and one cup.
Teacher: The way I want you to read it is one cup and zero. Read me what you have.
Students: One cup and zero.
Teacher: Plus one, again. Remember, the plus one means one square, not one cup.

(illustration 12-1-7)
(One cup and one square correctly placed on the trading board.)

Teacher: Check your neighbor and have your neighbor check you. Read me the number of cups and squares you have on your trading board.
Students: One cup and one.
Teacher: Plus one.

(illustration 12-1-8)
(One cup and two squares correctly placed on the trading board.)

Teacher: Check your neighbor and have your neighbor check you. Read me the number of cups and squares you have on your trading board.
Students: One cup and two.
Teacher: Plus one.

(illustration 12-1-9)
(One cup and three squares correctly placed on the trading board.)

Teacher: Check your neighbor and have your neighbor check you. Read me the number of cups and squares you have on your trading board.
Students: One cup and three.
Teacher: Plus one. See if you can remember what the rules say to do for base four. Remember, in base four, we cannot have any groups of four lying around loose. Check your neighbor and have your neighbor check you to see if you agree on what the base-four rules say to do.

(illustration 12-1-10)
(Two cups and zero squares correctly placed on the trading board.)

Teacher: Read me the number of cups and squares you have on your trading board.
Students: Two cups and zero.
Teacher: Plus one. Read me what you have.
Students: Two cups and one.

The teacher continues saying plus one as the students add squares and cups to their trading boards until the students reach three cups and three.

(illustration 12-1-11)
(Three cups and three squares correctly placed on the trading board.)

Teacher: Check your neighbor and have your neighbor check you to see that we all have the same number of cups and squares on our trading boards.
Now, we're going to count down instead of up. What do you think I want you to do when I say minus one?
Student: Take away one.
Teacher: One what?
Student: One square.
Teacher: Okay. Minus one.

(illustration 12-1-12)
(Three cups and two squares correctly placed on the trading board.)

Teacher: Check your neighbor to see if you have figured out what I want minus one to mean. Read me the number of cups and squares on your trading board.
Students: Three cups and two.
Teacher: Minus one.

(illustration 12-1-13)
(Three cups and one square correctly placed on the trading board.)

Teacher: Check your neighbor and have your neighbor check you. Read me your trading board.

Students: Three cups and one.
Teacher: Minus one.

(illustration 12-1-14)
(Three cups and zero squares correctly placed on the trading board.)

Teacher: Check your neighbor and have your neighbor check you. Read me your trading board.
Students: Three cups and zero.
Teacher: Minus one. What do you think the rules for bases would tell you to do now?
Student: You take away a cup, so you only have two cups left.
Student: No, you can't take anything away because you are out of squares on the square side and you said we were only taking away squares.
Student: No, you take a square out of a cup, so you still have three cups left.

The teacher allows the students to puzzle over how to take one square away from the three cups and zero squares. If the students decide correctly how to get more squares into the square column, then the minus one process continues. If the students collectively do not know the correct procedure, the teacher provides the rule.

Teacher: When we are subtracting one square each time and we run out of squares on the square side of our trading board, we dump the squares out of one cup to get some more.
Student: But you said we couldn't have any loose groups of four.
Teacher: True. So we can only dump out a cup of squares if we already need to take a square away. As soon as we dump out the squares we have to take the one square away immediately.
Student: But why not just take one square out of a cup and leave all the other squares still in?
Teacher: How many squares did we have to have to make a cup?
Students: Four.
Teacher: If we take a square out of a cup, then there won't be four in the cup any more. The rule for the cups in base four is that each of the cups must have four squares inside of it. Not five. Not three. Not any other number. Just four.
This is what I get when I minus one from three cups and zero.

(illustration 12-1-15)
(A three step illustration. First step: Three cups and zero squares correctly placed on the trading board. Second step: Two cups and four squares correctly placed on the trading board. Third step: Two cups and <u>three</u> squares correctly placed <u>on</u> the trading board, with <u>one</u> square slid <u>off</u> the board but still visible.)

Teacher: Check your neighbor and have your neighbor check you. Read me your trading board.
Students: Two cups and three.
Teacher: Minus one.

The teacher continues saying minus one until the students reach zero cups and zero on their trading boards. As the students place the squares and cups on their trading boards for plus and minus one, the teacher roams around the room judging how well the class collectively understands its first lesson in base four.

A sequence of events...

A printed sequence of events implies a particular sequence to follow. But how can any book know our students better than we do? The sequence that we should follow is the one that matches our students best. The sequence that we follow should be the sequence that we set.

Different bases are everywhere around us. But that fact alone is not sufficient reason to teach bases to our class. Our promise to our students is that we will leave no one behind. Everyone learns in our class, no exceptions, no excuses. Teaching different bases allows us to meet our promise to each child.

As we teach the plus and minus lesson in base four, we observe how well each child understands. On the next day, we repeat the plus and minus lesson in a different base and give everyone who did not understand a second chance. If some students do not grasp the concept by day two, we use day three to teach the lesson once again. If it takes a fourth day, then we teach it again on day four. With no pre-established sequence to follow, we repeat the plus and minus lesson using different bases until every child understands. Teaching many bases takes away the pressure to move some students too quickly ahead while leaving others too quickly behind.

As long as some students understood the lesson on day one, the lesson on day two begins with more "teachers" than were available on day one, because the students who understand in one base, will understand for the next. Each new day that we teach the plus and minus lesson, more students catch on to different bases and become available to help the dwindling number of students who do not. Teaching different bases allows all our students to help each other learn.

Teaching different bases also helps our students understand the structure of numbers in a way not possible if we only learn base ten. Mathematics is patterns and connections. By seeing the patterns from one base to another, our students learn the basic structure of numbers, regardless of the base.

Plus and minus with recording...

Teacher: We will use base four. Start with zero cups and zero squares on your trading board. Read me what you have.
Students: Zero cups and zero.
Teacher: Now we are going to record the cups and squares for plus one on a recording strip. Please draw a cup and a square at the top of your recording strip like this.
Read me what you have on your trading board again.
Students: Zero cups and zero.
Teacher: We record zero cups and zero like this.

(illustration 12-1-16)
(Recording strip with a cup and a square drawn at the top. Cup on the left, square on the right. Zeros are written in the spaces beneath the cup and square drawings.)

Teacher: Please record zero cups and zero squares on your recording strip, too.
Plus one. Read me what you have.
Students: One.
Teacher: That's true, but the way I want you to read it now is to tell me how many cups you have, too. How many cups do you have?
Students: Zero.
Teacher: So, you read that as zero cups and one. Please read me what you have on your trading board.
Students: Zero cups and one.
Teacher: We record zero cups and one like this.

(illustration 12-1-17)
(The recording strip with zero and one written in the spaces beneath the two previous zeros.)

Teacher: Please record zero cups and one on your recording strip, too.
Plus one.

The plus-one game continues all the way up to three cups and three. As the students record their numbers, the teacher asks them to examine the columns of figures on their recording strips for patterns.

(illustration 12-1-18)
(Place the recording strip adjacent to the teacher's questions.)

0	0
0	1
0	2
0	3
1	0
1	1
1	2
1	3
2	0
2	1
2	2
2	3
3	0
3	1
3	2
3	3

Teacher: Look down the squares column on your recording strip. Can you see any number patterns that might help you know what numbers will come next?
How many numbers are between each zero in the squares column?
What base is this?
Are there any patterns for the numbers in the cups column that tell you what numbers to expect?
How many zeros altogether are in the cup column?
How many ones? Twos? Threes?
What base is this?

When the class reaches three cups and three squares, the minus-one game begins. The students start with a new recording strip and record the numbers back down to zero cups and zero squares.

(illustration 12-1-19)
(Place the recording strip adjacent to the teacher's questions.)

3	3
3	2
3	1
3	0
2	3
2	2
2	1
2	0
1	3
1	2
1	1
1	0
0	3
0	2
0	1
0	0

Teacher: Are the patterns you can see for minus one the same as or different than the patterns you saw for plus one?
Are there any number patterns that help you know what numbers will come next?

Still the same?...

On successive days, we record plus and minus one in base five and then in base six. The lesson is the same. If the questions that we ask are the same as well, are the answers still the same?

(illustration 12-1-20)
(Recording strips for bases five and six. In base five, the strip goes from 0,0 to 4,4. In base six, the strip goes from 0,0 to 5,5. Put the recording strips adjacent to the questions below.)

Teacher: Look down the squares column on your recording strip. Can you see any number patterns that might help you know what numbers will come next?
How many numbers are between each zero in the squares column?
What base is this?
Are there patterns for the numbers in the cups column that tell you what numbers come next?
How many zeros altogether are in the cup column?
How many ones? Twos? Threes? Fours? (And, for base six, fives?)
What base is this?
Are the patterns for minus one the same as or different than the patterns for plus one?
Are there any number patterns that might help you know what numbers will come next?

The patterns that exist within and between bases help us all make sense of numbers, including numbers in base ten. When numbers make sense to all of us, none of us is left behind.

Squares, cups and bowls...

Teacher: Today we will use a trading board with three columns instead of two.

(illustration 12-1-21)
(Three column trading board with a drawing of a bowl, cup and square at the top. Bowl on the left, cup in the middle, square on the right.)

Teacher: We will play the plus-one game again. Because we are learning how to use the new trading board, we won't write down any numbers at the start. Today's base is base four.
Plus one. Read me what you have.
Students: One.
Teacher: I would like you to read me the cups you have, too, even if you only have zero cups.
Students: Zero cups and one.
Teacher: Plus one. Read me what you have.
Students: Zero cups and two.

When the students are recording and looking for patterns on their recording strips, they keep to plus-one, or later on, plus-two. However, while they are learning to use their three-column trading boards, if the students have already demonstrated an understanding of plus and minus one for the two-column trading board, we may vary the lesson by not always confining ourselves to saying "plus one."

Teacher: Think about this. Plus two.
Don't read me what you have yet. Please check your neighbor and have your neighbor check you to see if you agree about what to do when I say plus two instead of plus one.
Teacher: Read me what you have.
Students: One cup and zero.
Teacher: Plus two again.

The teacher continues to say plus one (or two or three) until the students have reached three cups and three on their trading boards.

(illustration 12-1-22)
(Three-column trading board with three cups and three squares on it.)

Teacher: Plus one.
Students: Four cups and zero.
Teacher: Remember the rule for bases? The rule for bases is that we cannot have any groups of the base number lying around loose. In base four, we cannot have any groups of four left over. That means we cannot have four squares, or four cups, or four of anything lying around loose. So, we have to gather up the cups and put them in a bowl, like this.

(illustration 12-1-23)
(Three-column place-value board. One bowl with four cups in it. All the squares still in the cups.)

The teacher places the cups into the bowl with the squares still in them. Although it is numerically correct to dump the squares out of their cups when placing them into the bowl, the rule for this game is that the squares stay in the cups, so that neighbors can check neighbors and the teacher wandering around the room watching students work, can tell at a glance if students have gathered together the right number of cups to go into the bowl. When students play the minus-one game, the cups in the bowl come out again. If the squares are loose in the bowl and not in cups, what would the frame of reference be that students use to understand the number that results when a square is taken from a bowl?

Teacher: We read this as one bowl, zero cups and zero. Please read me what you have.
Students: One bowl, zero cups and zero.
Teacher: Plus one.

The plus-one game continues to three bowls, three cups and three.

Teacher: Check your neighbor and have your neighbor check you to see that we all have the same number of bowls, cups and squares on our trading boards. Now, minus one.

The minus-one (two, or three) game is essentially the same as the minus game the students have already played with cups and squares. Students play until they reach the point where they run out of squares and cups to take away and are left with only bowls.

(illustration 12-1-24)
(Three column trading board with three bowls, zero cups and zero.)

Teacher: Minus one. Read me what you have.
Student: We can't do it.
Teacher: But when you ran out of squares before, you got more squares from the cups.
Student: We don't have any squares or cups left.
Teacher: How can you get more cups?
Student: We can take a cup out of the bowl.
Teacher: The bowl in base four must always have four cups in it. If you take one cup out, you will have to take the other cups out as well. Let's try taking out the cups and see what happens.

(illustration 12-1-25)
(Four stages. First stage: The board with three bowls, zero cups and zero squares. Second stage: The board with two bowls, four cups and zero squares. Third stage: The board with two bowls, three cups and four squares. Fourth stage: The board with two bowls, Three cups and three squares, with one additional square visible along side the board.)

Teacher: Read me what you have.
Students: Two bowls, three cups and three.
Teacher: Minus one.

The teacher continues saying minus one (or two or three) all the way to zero bowls, zero cups and zero.

With recording...
Teacher: Now we are going to record the bowls, cups and squares for plus one on a recording strip. Please draw a bowl, a cup and a square at the top of your recording strip, like this.
Start with zero bowls, zero cups and zero squares on your trading board. Read me what you have.
Students: Zero bowls, zero cups and zero.
Teacher: We record zero bowls, zero cups and zero like this.

(illustration 12-1-26)
(Recording strip with a bowl, a cup and a square drawn at the top. Bowl on the left, cup in the middle, square on the right. Zeros written in the spaces beneath the bowl, cup and square drawing.)

Teacher: Please record zero bowls, zero cups and zero squares on your recording strip.
Plus one. Read me what you have.
Students: Zero bowls, zero cups and one.
Teacher: Please record zero bowls, zero cups and one on your recording strip, too. Plus one.

(illustration 12-1-27)
(Recording strip with zero, zero and one written in the spaces beneath the three previous zeros.)

Teacher: Plus one.

When the teacher thinks the students understand how to record plus one for three columns, the students record the numbers on their own. As the students record their numbers, the teacher asks them to examine the columns of figures on their recording strips for patterns. The questions are familiar. Will the answers also be?

(illustration 12-1-28)
(A recording strip with the numbers from 0,0,0 through 3,3,3 on it. Place the recording strip adjacent to the teacher's questions.)

Teacher: Look down the squares column on your recording strip. Can you see any number patterns that might help you know what numbers will come next?
How many numbers are between each zero in the squares column?
What base is this?
Are there any patterns for the numbers in the cups column that tell you what numbers to expect?
How many zeros altogether are in the cup column?
How many ones? Twos? Threes?
What base is this?
Are there any patterns for the numbers in the bowls column that tell you what numbers to expect?
How many zeros altogether are in the bowl column?
How many ones? Twos? Threes?
What base is this?
Is there any connection between the number of zeros, ones, twos and threes in the bowls column and the number for the base?

When the students reach three bowls, three cups and three, they record minus ones back down to zero, zero, zero.

Teacher: Are the patterns you can see for minus one the same as or different than the patterns you saw for plus one?
Are there any number patterns that help you know what numbers will come next?

On successive days, we record plus and minus one in base five, then in base six, then in base three.

(illustration 12-1-29)
(Three column recording strips for bases three, five and six. In base three, the strip goes from 0,0,0 to 2,2,2. In base five, the strip goes from 0,0,0 to 4,4,4. In base six, the strip goes from 0,0 to 5,5,5.)

If the questions that we ask are the same for each new base, are the answers still the same, as well?

Base ten...

When our students can see the patterns for plus and minus one in different bases, we ask them to see the patterns once again in base ten.

Teacher: Today we will play the plus-one game again. Today's base is base ten.
Please start with zero bowls, zero cups and zero squares on your trading board and, as you play plus one for base ten, record your bowls, cups and squares on a recording strip.

As students record their numbers, the teacher asks them to examine the columns of figures on their recording strips for patterns and then asks the familiar questions.

(illustration 12-1-30)
(A recording strip, broken in the middle so it will fit on the page, with the numbers from 0,0,0 through 1,1,1 on it. Place the recording strip adjacent to the teacher's questions.)

Teacher: Look down the squares column on your recording strip. Can you see any number patterns that might help you know what numbers will come next?
How many numbers are between each zero in the squares column?
What base is this?
Are there any patterns for the numbers in the cups column that tell you what numbers to expect?
How many zeros altogether are in the cup column?
How many ones? Twos? Threes? Fours? Fives?...
What base is this?
Are there any patterns for the numbers in the bowls column that tell you what numbers to expect?
How many zeros altogether are in the bowl column?
How many ones do you think would be in the bowls column if you kept adding to your recording? How many twos? Threes? Fours? Fives?
What base is this?
Is there any connection between the number for the bowls column and the number for the base?

Looking for patterns in base ten is not a new activity for our students. As we reviewed the counting numbers at the start of the school year, we looked for patterns in the numbers from 1 to 100 in base ten, though we did not call the numbers base ten. A 0-100 counting strip and 0-99 number chart may still be on our wall from pattern searches the class has previously done.

We have seen the base-ten patterns before, but we did not then have the chance to see the base-ten patterns as patterns in a broader stream. Mathematics is patterns and connections. What we see in bases three and four and five, we see in base ten as well.

Teaching different bases allows us to repeat lessons without having the lessons seem repeated. Teaching different bases also allows us to understand the structure of the numbers that we use regardless of what the base might be.

Lesson Two

Purpose	Learn about adding or subtracting numbers greater than one in different bases.
Summary	Students play racing-up and racing-back place-value trading games.

Materials	Trading boards, squares, cups, bowls, dice.
Topic	Racing up in base 4, winner is the first up.
Topic	Racing up in base 5, winner is the last up.
Topic	Racing up in base 6, winner is the first up.
Topic	Racing up in base 3, winner is the last up.
Topic	Racing up in base ten, winner is either the first or last up, decided in advance.
Topic	Racing back in base 4, winner is the first back.
Topic	Racing back in base 5, winner is the last back.
Topic	Racing back in base 6, winner is the first back.
Topic	Racing back in base 3, winner is the last back.
Topic	Racing back in base ten, winner is either the first or last up, decided in advance.

Racing up...

Teacher: Today I am going to teach you a game that uses your squares, cups and bowls. We call this game "Racing Up." I'll teach Kyle, Aaron and Ashley to play. Everyone else please watch so you can learn how to play, too.
Kyle, you be the banker. The banker keeps track of all the squares.
Today we will be playing in base four. How many squares go in each cup in base four?
Students: Four.
Teacher: How many cups go in to a bowl in base four?
Students: Four.
Teacher: The goal is to be the first one to reach a bowl. Aaron, please roll the die first.

(illustration 12-2-1)
(Two trading boards. Aaron rolling a die. Have the die roll match the dialog.)

Teacher: Aaron, what did you roll?
Aaron: A three.

The dice the students are using for this game, and for all their other dice-rolling experiences with different bases, are not the usual one-through-six numbered dice. The base-three dice have the numbers 0, 1 and 2 on them. The base-four dice have 0, 1, 2 and 3. The base-five dice have 0, 1, 2, 3 and 4. The base-six dice have 0, 1, 2, 3, 4 and 5. For rolling base-ten numbers, we use the dice from bases five and six.

(illustration 12-2-2)
(Dice for bases three, four, five and six. Show all the sides. The numbers on the dice are written on with colored marking pen. Include in the caption that the different colors are used to tell which dice go with which base. Include, also, that these particular dice have been made by writing on blank one inch wooden cubes.)

Teacher: How many squares do you think the rules for this game will tell the banker to give Aaron to put on his trading board?
Students: Three.
Teacher: Banker Kyle, please give Aaron three squares. Aaron, please put the squares on your trading board.

(illustration 12-2-3)
(One of the trading boards with three squares on it.)

Teacher: Ashley's turn. Ashley's turn cannot start until Aaron has placed his squares on his trading board. Ashley, please roll the die. What did you roll?
Ashley: A one.
Teacher: How many squares do you think the rules for this game will tell the banker to give Ashley to put on her trading board?
Students: One.
Teacher: Banker, please give Ashley one square to put on her trading board.

(illustration 12-2-4)
(Both trading boards visible. One with three squares on it, one with one square on it.)

Teacher: When does Aaron get his next turn?

Students: When Ashley has put her squares on her trading board.
Teacher: Correct. Aaron, please roll the die.

(illustration 12-2-5)
(Pattern the illustration after the illustration on page 90 in *Math... a Way of Thinking*. Base four instead of base five. Squares replace beans. The trading board depicted has a larger section for bowls than it does for squares.)

The banker's position is an important one. The student rolls the die and the banker issues the squares. The banker makes sure that only the number of squares rolled is the number of squares that appear on the trading board. The banker also reminds the players when squares turn into cups and cups turn into bowls.

Once the students have seen the game played, they divide into groups of three and begin playing for themselves. If the class does not divide evenly into threes, then four students can play together. Every time there is a winner, the loser trades places with the banker, and the game begins again.

As the students play, the teacher wanders around the room observing how well each student understands the process of moving from squares to cups and from cups to bowls.

First up, last up...

Learning is not a competition, but competition is a natural part of life. Children compete with each other all the time. They hold their breath, they stare at each other without blinking, they see how far they can spit, they engage in countless other contests, just to see who will win.

Competition in learning teaches students who learn at a different rate that their rate is not good enough. This is not the kind of competition we want to foster in our class. But competition in a well-controlled game can give all our students the opportunity to feel what it is like to win.

The first day that students play the racing-up game, the <u>first</u> child in each group to reach a bowl is the winner. If we want to, we can make the winner the first to reach two bowls, or one bowl, one cup, one square. The winner can race up to whatever goal we set. Once a winner is declared, the game starts over again with the trading boards cleared and a new banker to issue the squares. On the second day of the racing-up game, the <u>last</u> child in each group to reach a bowl is the winner.

Some students feel inside themselves that they can never be winners. Some adults among us feel the same way. Evidence of never winning can grow every time the never-winning child races to be first and loses once again. But when we alternate the first-up winning with last-up winning, we can chip away at the never-winning image inside. Instead of thinking "I never win," each losing child can think instead, "If we were playing the other set of rules today, I'd be winning now." The first-up, last-up changes may not eliminate the feeling some children have that they were born to lose, but the first-up, last-up changes do give everyone the chance to feel that they can win.

The message goes beyond the numbers on the dice and the squares on the board. *More* does not always have to mean better. *Less* can be better, too. Last up. First up. Either way can win.

First-up, last-up has a mathematical message, too. All the dice for all the bases have zeros written on them. When racing to be the first up to the bowl, a zero on a die is not the favorite roll. No progress is made toward a bowl with zero squares. When racing to be the last up to the bowl, a zero on a die is the favorite roll. No progress is made toward a bowl with zero squares. Zero's mathematical meaning is the same in either case, but what the meaning feels like changes according to the game.

Changing bases...

As our students race up to be the first to reach the big bowl in base four, we observe how well each child understands the racing game. On the second day, our students race to be the last up to the big bowl in base five. The lesson is repeated in a different base to give everyone who does not fully understand, a chance to learn from those who do. The lesson is also repeated so that we have time enough to watch the play of each and every child in our class.

Our assessment of the racing-up game is as simple and basic and straightforward as the math we teach. We watch to see who knows when to put the squares in a cup and who knows when cups become a bowl. If we feel our students would benefit from playing on another day, we change the base and begin the race again. There is no pre-established sequence we must follow. We repeat the racing game until our students show us that they are ready to move on. We play until everyone understands.

We change the bases so our students also learn that:

They can work in any base.
The number of the base does not change what numbers do.
There is always something new for everyone to learn.

When we know our students understand the racing game in different bases, we have them play the game again in base ten. What is true for base three or four or five is also true for ten. Numbers are numbers regardless of the base.

In base ten, each team of students racing up rolls one base-five and one base-six die. The base-five die has all the numbers from 0 through 4. The base-six die has all the numbers from 0 through 5. The numbers on the dice are added to find the number of squares to put on the trading boards. Together, the dice produce the number combinations all the way to nine.

Racing back...

(illustration 12-2-6)
(Two trading boards, each has one bowl, one cup and one square on it. The bowls and cups are in base four.)

Teacher: Today I am going to teach you a different game for your squares, cups and bowls. We call this game "Racing Back." I'll teach Hanna, Monica and Kevin to play. Everyone else please watch so you'll be ready to play when your turn comes. Everybody please read me what I have on each trading board.
Students: One bowl, one cup and one.
Teacher: What base is this?
Students: Four.
Teacher: Okay. Hanna, you will be the banker. The goal of the game is to be the first one to reach zero bowls, zero cups and zero squares. Monica, please roll the die first. What did you roll?
Monica: A two.
Teacher: How many squares do you think the rules will tell Monica to give to the banker?
Students: Two.
Teacher: Monica, do you think you can give Hanna two?
Monica: Yes.

(illustration 12-2-7)
(Illustration in three stages. First stage: Trading board with one bowl, one cup and one square, and a die next to it with the number two showing. Second stage: The squares from the one cup in the cup column dumped into the squares column, with the cup removed. Third stage: Two squares gone. Final board had one bowl, no cups and three squares.)

What Monica has done so far is essentially the same as what every student has learned to do while playing the minus-one game in several different bases. Monica and everyone else know that when not enough squares are in the square column, she can dump squares out of a cup to get more.

Teacher: Kevin's turn. Kevin, please roll the die. What did you roll?
Kevin: A three.
Teacher: How many squares do the rules for this game tell Kevin to give to the banker?
Students: Three.
Teacher: Kevin, do you think you can give Hanna three?
Kevin: Yes.

(illustration 12-2-8)
(Illustration in three stages. First stage: Trading board with one bowl, one cup and one square, and a die next to it with the number three showing. Second stage: The squares from the one cup in the cup column dumped into the squares column, with the cup removed. Third stage: Three squares gone. Final board had one bowl, no cups and two squares.)

Teacher: Monica's turn. Monica, please roll the die. What did you roll?
Monica: A three.

(illustration 12-2-9)
(Monica's trading board with one bowl, no cups and no squares on it. Three squares visible off the board.)

Teacher: Kevin's turn. Kevin, please roll the die. What did you get?

Kevin: One.

(illustration 12-2-10)
(Kevin's trading board with one bowl, no cups and one square on it. One square visible off the board.)

Teacher: Monica, please roll the die. What did you get?
Monica: A two.
Teacher: Monica, how can you take two squares away when you don't have any squares and you don't have any cups either?

(illustration 12-2-11)
(Illustration in four stages. First stage: Trading board with one bowl, no cups and no squares, and a die next to it with the number two showing. Second stage: The four cups from the one bowl in the bowl column placed in the cups column, with the bowl removed. Third stage: The squares from one cup dumped into the squares column and the cup removed. Fourth stage: Two squares gone. Final board has zero bowls, three cups and two squares.)

Playing the minus-one game in Lesson One showed Monica and everyone else how to get squares from cups. Playing the minus-one game also showed Monica and everyone else how to get squares from a bowl when no more cups were in the cup column. Monica knows that when not enough squares are in the square column, she must take cups from a bowl so that she has cups for the squares she needs. The learning we do one day connects to the learning we do the next.

(illustration 12-2-12)
(Pattern the illustration after the illustration for racing up on page 90 in *Math... a Way of Thinking*, except the illustration will now be for racing down to zero, zero, zero. Base four instead of base five. Squares replace beans. The trading board depicted has a larger section for bowls than it does for squares.)

When our students race up, each new roll of the die tells them the number of squares to add to their board. Squares become cups. Cups become bowls. For racing up, change comes one step at a time.

Racing back is not as simple as racing up. Students take squares from the squares they have on their boards until there are no more squares to take. Then the students dump squares from cups one cup at a time, until there are no more cups to take. When there are no more cups to take, the students must get more cups from a bowl.

Changing bases again...

As our students race to be the first to reach zero bowls, zero cups and zero squares in base four, we wander around the room observing how well each child understands the game. On the second day, our students race to be the last one down to zero, zero, zero in base five. The lesson is repeated so that everyone who does not fully understand can learn from everyone who does.

When our students understand the racing game in different bases, we have them play the game again in base ten. We use the same base-five and base-six dice we used for racing up for racing back.

Lesson Three

Purpose	Learn the concept of place value.
Summary	Students learn a game that teaches them the value places have.
Materials	Trading boards, squares, cups, bowls, recording paper, dice.
Topic	Place value game in base 4, most wins.
Topic	Place value game in base 5, least wins.
Topic	Place value game in base 6, most wins.
Topic	Place value game in base 3, least wins.
Topic	Place value game in base ten, either most or least wins, decided in advance.

The Place Value Game...

Teacher: Today we will play a new game. This game is called "The Place Value Game." For today, we will be playing in base four. Brenda and Debbie, please help me demonstrate how to play.
When we were racing up to a bowl or back down again, each team of players had only one die. For this game, each group will need three dice. Brenda, please roll the three dice.

(illustration 12-3-1)
(Three dice, showing the numbers 3, 0, 2. The dice are placed next to a blank trading board.)

Teacher: The object of this game is to arrange your three dice on the trading board in a way that gives you the maximum number of total squares. Brenda, put your dice on the trading board, one die in each column.

(illustration 12-3-2)
(The 3 die in the bowl column, the 0 die in the cup column and the 2 die in the square column.)

Teacher: Okay, now we are going to put out the same number of bowls, cups and squares as the dice tell us to. How many squares in the square column?
Students: Two.
Teacher: How many cups in the cup column?
Students: None.
Teacher: How many bowls in the bowl column?
Students: Three.
Teacher: Brenda and Debbie, please help me make three bowls worth of squares.
Student: Let's just put out the three bowls with no squares in them.
Teacher: The rule for this game is that the bowls must have the right number of cups and squares in them. But after the bowls are made for one game, you don't have to remake them for the next time it's your turn to roll the dice.

(illustration 12-3-3)
(The trading board with the 2 die removed and two squares in the square column, the 0 die removed and no cups added, and the 3 die removed and three bowls in the bowl column.)

Teacher: How many squares does Brenda have on the trading board?
Student: Two.
Teacher: I mean, how many squares does Brenda have altogether in the bowls and cups and squares columns?
Students: ?
Teacher: How many squares does Brenda have in the squares column?
Students: Two.
Teacher: How many squares in the cups column?
Students: She doesn't have any.
Teacher: So, that's zero. How many squares in the bowls column?
Student: Three bowls' worth.
Teacher: How many squares is that?
Student: Have Brenda count them and see.
Teacher: How many cups in a bowl in base four?
Students: Four.
Teacher: How many squares in one cup for base four?
Students: Four.
Teacher: How many squares in two cups?
Students: Eight.
Teacher: How many squares in three cups?
Students: Twelve.
Teacher: How many squares in four cups?
Students: Sixteen.
Teacher: Then how many squares are there in one bowl if the bowl has four cups in it?
Students: Sixteen.
Teacher: If there are sixteen squares in one bowl, how many squares are there in three bowls? You may use your calculators to help you if you wish.
Students: Forty-eight.
Teacher: So there are forty-eight squares in the bowls column and two squares in the squares column. How many squares does Brenda have altogether?
Students: Fifty.
Teacher: Debbie, please roll the three dice and put them on the trading board, one die in each column. Arrange your three dice to get the maximum number of squares.

(illustration 12-3-4)
(Three dice, showing the numbers 2, 1, 2. The 2 die in the bowl column, the 2 die in the cup column and the 1 die in the square column.)

Teacher: What do we do now?

Students: Put out the number of bowls, cups and squares that Debbie's dice say to.

Teacher: How many squares in the square column?

Students: One.

Teacher: How many cups in the cup column?

Students: Two.

Teacher: Debbie and Brenda, please help me make two cups of squares.

Teacher: How many bowls in the bowl column?

Students: Two.

Teacher: Debbie and Brenda, please help me make two bowls of squares.

Student: Can't Debbie just use two of Brenda's from before?

Teacher: I said that after the bowls of squares are made for one game, you don't have to remake them for the next time it's your turn to roll. The game isn't over until both Debbie and Brenda have put the bowls, cups and squares on their trading boards, so we can't take any of Brenda's bowls and give them to Debbie.

(illustration 12-3-5)
(The trading board with the 2 die removed and two bowls in the bowl column, the 1 die removed and one square in the square column, and the 2 die removed and two squares in the square column.)

Teacher: How many squares does Debbie have in the bowls, cups and squares columns? You may work with a partner to figure out the total and you may use your calculators to help you if you wish. When you think you have an answer for the total number, write your answer on your chalkboard and put your chalkboard face down on your desk.

If the class collectively finds the total, then the students are almost ready to begin playing on their own. If the class collectively finds something else, the teacher leads the students once again through the process of counting all the squares.

Students: (On their chalkboards) Thirty-four.

Teacher: How many squares did Brenda have altogether?

Students: Fifty.

Teacher: Then who had more total squares, Brenda or Debbie?

Students: Brenda.

Teacher: Then Brenda is the winner. All you need to know before you start playing against each other is how to record your scores. How many bowls, cups and squares does Brenda have?

Students: Three bowls, no cups and two squares.

302

Teacher: How many bowls, cups and squares does Debbie have?

Students: Two bowls, two cups and one square.

302 221

Teacher: Which of these two sets of number makes the highest total number of squares?

Students: Brenda's.

Teacher: This is the symbol mathematicians use to indicate which number is bigger.

302 › 221

Actually, the same symbol for the bigger number is also used for the littler number. The big, open end points to the bigger number and the little, pointed end points to the littler number. How we read the symbol depends on which way it is pointing. The way I have put it now it says "is more than" or "is greater than." If I wrote Debbie's number first and Brenda's second, I would use the symbol for "is less than," like this:

221 ‹ 302

If you think you are ready, you can start rolling your dice and keeping track of who wins for each of the games you play.

Student: Can we put our dice out in any order we want?

Teacher: Yes. Roll all three dice at once and then put them in any order you wish.

If no student asks, we tell our students to put the dice out in any order they wish. The purpose of the place-value game is show our students that the value of a numeral changes depending on its placement. A 3 in the squares column does not have the same value as a 3 in the cup column, or a 3 in the bowls column. Each place has a value all its own.

As our students play, we walk around the room observing whether our students know that 321 is a better arrangement than 231 or 132. Brenda won the demonstration game, but Debbie was the one who actually placed her dice in the best possible order. We assess who understands that place has a value by observing how students arrange the dice on their boards.

We ask our students to record the results of each game on paper. Recording helps each student make the connection between symbols on paper and the numbers of squares these symbols represent. Concept, connecting, symbolic. Making connections is a part of math.

Changing bases, changing rules...

Teacher: Today we will play the place-value game again, only this time we'll play in base five. Yesterday, the winner was the person with more squares. Who do you think I will have today's winner be?

Students: The one with less.

In the number 444, are all the fours equal? We know that they are not, but we teach our students what we know about number by letting them play place value games as we change the base. When our students understand place value, there is one more base we ask them to learn.

Teacher: Today we will be playing the place value game in base ten.

Lesson Four

Purpose	Learn to create addition and subtraction problems in any base three through ten.
Summary	Students learn how to create addition and subtraction problems and how to find and check the answers to the problems they create.
Materials	Trading boards, squares, cups, bowls, addition and subtraction recording paper blackline, dice, calculators.
Topic	Addition problems are created in base 4.
Topic	Addition problems are created in base 5.
Topic	Addition problems in any base 3 to 6.
Topic	Subtraction problems are created in base 4.
Topic	Subtraction problems are created in base 5.
Topic	Subtraction problems in any base 3 to 6.
Topic	Addition problems are created in base ten.
Topic	Subtraction problems are created in base ten.
Homework	If students need more practice, dice and cups and bowls can be sent home, but only with children who understand the basic process.

Adding...

Teacher: Today we will be creating addition problems in base four. You will each need to make a new trading board to record the problems we create.

(illustration 12-4-1)
(An example of an addition trading board. The example is patterned after the illustrations on page 95 of Mathematics... a Way of Thinking. For this illustration, however, there are only three columns on the board. Bowls, cups, squares.)

Teacher: I will give you an example of how to make up addition problems before you begin creating your own. First, I roll two base four-dice to see what numbers to put in the top row of my trading board. Then I put the dice on the top row. Then I record the numbers I am putting on my trading board.

(illustration 12-4-2)
(Trading board with a 2 die in top space in the cups column and a 3 die in the top space in the squares column. 23 written next to (and not on) the trading board.)

Teacher: Now, in place of the dice, I put out the cups and the squares for the numbers I have rolled. How many squares go in the squares column?
Students: Three.
Teacher: How many cups?
Students: Two.

(illustration 12-4-3)
(Trading board from above illustration with the dice replaced by two cups and three squares. The 23 is still visible alongside the trading board.)

Teacher: Now, I roll my two dice to see what numbers to put in the second row of my trading board. Then I put the dice on the second row. Then I record the numbers.

(illustration 12-4-4)
(A 3 die in the second space in the cups column and a 1 die in the second space in the squares column. 31 written beneath the 23, that is already next to the trading board.)

Teacher: Now, in place of the dice, I put out the cups and the squares for the numbers I have rolled. How many squares go in the squares column?
Students: one.
Teacher: How many cups?
Students: Three.

(illustration 12-4-5)
(The illustration from above with the dice replaced by three cups and one square.)

Teacher: We could add more rows of cups and squares to our problem, but for now we will stop at two rows. So that I can tell that I am done adding rows to my trading board, I draw a line underneath the numbers I have recorded.

$$\begin{array}{r} \mathbf{23} \\ \underline{\mathbf{31}} \end{array}$$

Once we have finished recording the numbers for the cups and squares, we slide all the cups and squares together at the bottom of the trading board and see what exchanges we can make.

(illustration 12-4-6)
(The three steps in adding the squares and cups on the trading board. First, slide all the cups and squares to the bottom of the board. Second, put the four squares in a cup and move the cup to the cup column. Third, put four of the cups in a bowl and move the bowl to the bowl column.)

Teacher: Once we have made all the exchanges we can, we write our answer beneath the numbers we already recorded.

$$\begin{array}{r} \mathbf{23} \\ \underline{\mathbf{31}} \\ \mathbf{120} \end{array}$$

Now, we'll make up another problem, only this time, you tell me how to do it. What do I do first?
Students: Roll two dice...

When our students collectively understand the process of using dice to make up addition problems in base four, we give them time to create their own problems to solve. Just as we did for *Beginning Addition and Subtraction,* we give our students numbers of minutes to work and not numbers of problems or numbers of pages to complete.

(illustration 12-4-7)
(The blackline master for recording addition problems with bowls, cups and squares. The blackline master has a place for the student's name and for the base number in which the student is working. Also, show a recording sheet filled in for base four addition.)

Although our students may record their work on regular paper, we may choose to give them recording paper so they can practice the art of writing the numbers in columns. Or, we can simply talk to our students about the proper placement of the numbers as they record. We use the way that works best for us.

The answers flow from the materials...

As our students work, we wander around the room checking the numbers they record. We look to see if:

The numbers recorded on paper match the cups and squares on the trading board.
All the cups and squares are added together.
All the exchanges are made according to the rule for bases.

We do not keep what we are looking for a secret. We teach our students to check their work themselves. The answers flow from the materials our students use.

Teacher: As I walk around the room looking at the problems you are creating for addition in base four, I am looking for fours or fives or sixes on your papers. Why do you think I am looking for fours and fives and sixes?
Student: We aren't supposed to have any fours or fives or sixes.
Teacher: Then why would I be looking for those numbers?
Student: To see if we got any problems wrong?
Teacher: To see if you forgot the rule for bases. If I see a four or some larger numeral on your paper, that means you forgot the rule.
You can look at your own paper to see if you have any fours or fives or sixes. You can also check your neighbor and have your neighbor check you to see what numbers you have written down.

If we do not have the time to visit every student, we can assess our students' learning with a problem out the door. (*Beginning Addition and Subtraction* page 000.)

Can I do it in my head?...

Student: Can I do it in my head?
Teacher: Do what in your head?
Student. The problems. Do I have to put the squares out each time?
Teacher: I personally prefer using the squares and the cups, but you may work the problems in your head as long as you can prove your answer to me with cups and squares if I ask you.

We do not encourage or reward the students who feel they are ready to abandon the materials. We value the answer, not how the answer is obtained. The students who think they are abandoning the materials are still using squares, cups and bowls inside their heads to envision the exchanges. We do not want any students to feel pressured to give up the security of the squares and cups prematurely, before the squares and cups are completely inside their minds.

Not every child speaks equally well, but all learn to speak. Not every child internalizes number concepts as quickly as all others, but all internalize eventually. We set up the environment for learning and then we give the learning time to take place.

Smaller numbers...

base 3	base 4	base 5	base 6	base 10
121	132	214	514	985
112	223	144	352	654

There are many reasons for teaching different bases. That smaller bases have smaller numbers is a most important one.

Bases three, four, five and six, have no numerals larger than can be counted out on the fingers of one hand. When we teach addition in base ten, some of our students will be overwhelmed with the difficulty of adding 9 + 8. When we use the smaller numbers of the smaller bases, we shift the focus of the problems from remembering all the number facts to understanding what it means to add. We make it possible for all our students, regardless of their facility with sevens, eights and nines, to have success with smaller numbers.

Mixing bases...

When we introduced bases, we changed the base our students used each day. We may begin by giving each base its own day, but where we start is not where we end.

Teacher: Today I want you to make up addition problems for base five. When you finish one page, you may decide which base you wish to use for the problems on your next page. You may

choose any base from three to six. Please make sure that you write the base number that you are working in at the top of your paper.
Student: Can we still keep working in base five?
Teacher: Yes, you can choose any base between three and six. Five is between three and six.
Student: Can we do a different base for every problem on the page?
Teacher: No. I want you to do the same base for the whole page. It is too hard for me to check your papers as I come around the room if you have more than one base on the same page.

The teacher may choose to answer "Yes" to this last student question and add in another rule.

Teacher: Yes, but I will have to be able to tell what base you used for each different problem. You will have to write the base number by each different problem like this.

(illustration 12-4-8)
(Example of three problems in three different bases with the base number written by the problem. The correct form is to write "base #" below and to the right of the answer.)

Subtracting...

Teacher: Today we will be creating subtraction problems in base four. You will each need to make a new trading board to record the problems we create.

(illustration 12-4-9)
(An example of a subtraction trading board. The example is patterned after the illustrations on page 98 of *Math... a Way of Thinking*. For this illustration, however, there are only three columns on the board. Bowls, cups, squares.)

Teacher: I will give you an example of how to make up subtraction problems before you begin creating your own. First, I put a bowl on my trading board. All the problems you create today will start with at least one bowl. Then, I roll two base-four dice to see what numbers to put in the cups and squares columns of my trading board. Then I put the dice on the trading board. Then I record the numbers I am putting on my trading board.

(illustration 12-4-10)
(Trading board with a bowl in the bowl column, a 1 die in the cups column and a 3 die in the squares column. 113 written next to (and not on) the trading board.)

Teacher: Now, I put out the cups and the squares for the numbers I have rolled. How many squares go in the squares column?
Students: Three.
Teacher: How many cups?
Students: One.

(illustration 12-4-11)
(Trading board from above illustration with the dice replaced by one cup and three squares. The 113 is still visible along side the trading board.)

Teacher: Now, I roll my two dice to see what numbers to put beneath my trading board.

(illustration 12-4-12)
(Two dice reading 2,1. The dice are beneath and not on the trading board from the above illustration.)

Teacher: I record these numbers, too. So that I can tell that I have finished making up my problem, I draw a line underneath the numbers I have recorded. I also have to be able to tell that this is a subtraction problem and not another one for adding, so I put in a minus sign.

$$\begin{array}{r} 113 \\ \underline{-\ 21} \end{array}$$

Now, I take from my trading board the number of cups and squares my dice say to remove. Tell me what to do for each column.

(illustration 12-4-13)
(The steps in subtracting the squares and cups on the trading board. First, slide the one square represented by the number on the die off the trading board, and put it below the die. For this particular problem, no exchanges are necessary to take the one die away. Second take all the cups out of the bowl

and put the cups in the cup column. Third, slide the two cups represented by the number on the die below the die.)

Teacher: Once we have taken away the numbers on the dice, we write our answer beneath the numbers we already recorded. The answer is how many bowls, cups and squares we have left on the trading board.

$$\begin{array}{r} \mathbf{113} \\ \underline{\mathbf{-\ 21}} \\ \mathbf{32} \end{array}$$

Now, we'll make up another problem, only this time, you tell me how to do it. What do I do first?
Students: Roll two dice.
Teacher: There is something I do before that.
Student: Put one bowl on the trading board.
Teacher: Then what?
Students: Roll two dice.

To help our students remember to place the bottom number off their trading boards and not on, we have neighbors checking neighbors until this subtraction rule is a part of what every student knows.

When our students collectively understand the process of using dice to make up subtraction problems in base four, we give them time to create their own problems to solve. We give them paper for recording or a blackline master for their use.

(illustration 12-4-14)
(The blackline master for recording subtraction problems with bowls, cups and squares. The blackline master has a place for the student's name and for the base number in which the student is working. Show a recording sheet filled in for base four addition.)

The problems our students create by rolling their dice mix together borrowing and not borrowing. For each problem our students must decide if there is any regrouping to be done. We want our students to learn to look at the numbers and to think about what the numbers mean. We do not want the process of borrowing and carrying to become simply a matter of knowing what the lesson is today.

Teacher: As I walk around the room looking at the problems you are creating for subtraction in base four, I am looking for fours or fives or sixes on your papers, just as I did for adding.
Look at your own paper to see if you have any fours or fives. Also, check your neighbor and have your neighbor check you to see if you have remembered all the subtraction rules.

As our students work, we wander around the room checking the numbers they record. We look to see if:

The numbers recorded on each student's paper match the bowls, cups, squares and dice on and off the student's trading board.
All exchanges are made according to the rule for bases, and no unnecessary exchanges are made.
All the cups and squares taken away from the trading board match the numbers on the dice below.

Mixing bases again...

Teacher: Today I want you to make up subtraction problems for base five. When you finish one page, you may decide which base you wish to use for the problems on your next page. You may choose any base from three to six. Please make sure that you write the base number that you are using at the top of your paper.

A little bit of each...

Do we make sure every child in our class understands addition in different bases before we teach subtraction, or do we mix together a little bit of each?

The rules for adding and subtracting in bases are the same rules our students learned for plus and minus one, or for racing up and back, or for the place-value game. The rules for adding and subtracting are the rules our students have been using all along. If the rules they use are not new to them, then does it make any difference if they mix the problems they now solve? Do we separate our adding and subtracting skills in life?

If we wish to, we may help every student understand addition in all bases before we ask our students to make up problems for subtraction. If we wish to, we may teach addition on one day and subtraction on the next or alternate addition and subtraction by the week. The choice is ours. Whatever we decide, we teach until everyone is taught. The sequence that we set matches how our students learn.

Big bowls...

If we feel our students are ready for a little more challenge, we add big bowls to the problems they create.

(illustration 12-4-15)
(Show a collage of Big Bowl materials. For base four, show a cup with four squares in it, a bowl with four cups in it and a big bowl with four bowls in it. Show recording sheets for addition and subtraction with a column for big bowls added on. Also show trading boards for addition and subtraction with the extra column added.)

Adding in base ten...

Teacher: Today we will be creating addition problems in base ten. You may use the same trading board for base ten that you used for creating addition problems in the other bases. You already know how to create addition problems in bases, but in base ten, you need to know one more fact.
What was the biggest number on your dice for base four?
Students: Three.
Teacher: What was the biggest number for base five?
Students: Four?
Teacher: For base six?
Students: Five.
Teacher: What do you think would be the biggest number for base ten?
Students: Nine.

If our students are not collectively able to see the pattern for the numbers on the base dice, we can write the pattern on the overhead and have our students discuss the numbers that they see.

Base	Highest number on die
2	1
3	2
4	3
5	4
6	5
7	?
8	?
9	?
10	?

Teacher: The biggest number we would need in base ten is nine, but the highest number on our dice is five. So, we are going to have to roll two dice and add their numbers together to get a number for each column.
I will give you an example of how to make up a base-ten addition problem before you begin creating your own.

The process of creating problems in base ten is the same as creating problems in any other base, except that the numbers for each column on the trading board are determined one at a time and not all at once. One base five die and one base six die are rolled, the numbers added together, and that number of squares or cups are placed on the trading board. The number of squares and cups to be added is written next to the trading board The process is repeated for the second row. Once the numbers have been determined, the squares and cups are added and the answer is recorded as was done for all the bases that have gone before.

Once the teacher has demonstrated the process, the teacher gives the class a turn.

Teacher: Now, we'll make up another problem, only this time, you tell me how to do it. What do I do first?
Students: Roll two dice and add them together.

When our students collectively understand the process of using dice to make up addition problems in base ten, we give them time to create their own problems.