Calculators for checking...

Calculators are for checking our work. We cannot use our calculators to check problems for bases three or four or five. Our calculators only check the work we do when we are in base ten. Once we are in base ten, students may use their calculators to check the answers to every problem they create.

Questions we ask when calculators are in use:

Have you created any problems where the answer you found for bowls, cups and squares is not the same as your calculator's answer?
Why do you think your two answers do not agree?
Which of the two answers do you believe? Why?
How can you tell if your calculator answer is right if you don't have any squares or cups or bowls around to compare?

Calculators provide a means for our students to check their own answers, but this tool for checking does not replace the thinking that our students do.

Subtracting in base ten...

Teacher: Today we will be creating subtraction problems in base ten. The same trading board you used for subtracting in all the other bases is still good for base ten.
I will give you an example of how to make up a base-ten subtraction problem before you begin creating your own.

The process of creating subtraction problems in base ten is the same as creating problems in any other base, except that the numbers for each column on the trading board are determined one at a time and not all at once. One bowl is placed in the bowl column of the trading board. 1 is written in the bowls space on the recording sheet. One base five die and one base six die are rolled, the numbers added together, and that number of squares or cups are placed on the trading board. The number is written on the subtraction recording sheet. That many squares or cups are placed on the trading board. The numbers are written in the top row of the subtraction recording sheet.

The two dice are rolled and added again to find the numbers to subtract. The numbers found are written in the second row of the subtraction recording sheet. For smaller bases, the dice were left beneath the trading board as a reminder of the numbers to subtract. However, since the base ten dice arrangement does not permit the dice to be left, the numbers are recorded on the recording sheet immediately. Once the numbers have been found, the subtracting is the same as the subtracting for every other base.

Once the teacher has demonstrated the process, the teacher gives the class a turn.

Teacher: Now, we'll make up another problem. This time, you tell me how. What do I do first?
Students: Put one bowl on the trading board.

When our students collectively understand the process of using dice to make up subtraction problems in base ten, we give them time to create their own problems.

Learning from the lesson...

We plan our lessons, expecting the lessons to go well. But we know that lessons do not always go the way we plan them; sometimes lessons go wrong. When a lesson does go wrong, there is much to learn from the experience.

Teacher: Today we will be making a three-column trading board. Please fold your paper in thirds.

(illustration 12-4-16)
(Show the variety of different folds produced from this instruction. Add in the illustration caption that many of the children did not know what thirds meant. Those that did thought that thirds came from folding the paper three times. Show one child having torn the paper in thirds.)

Teacher: On the right column of your trading board, please draw a square. In the middle column, please draw a cup. In the left column, please draw a bowl.

(illustration 12-4-17)
(Show examples of squares, cups and bowls drawn in places other than the top of the paper. Show an example of a child who drew the square, cup and bowl all in the same column, with the cup in the middle of the column. Also show examples of students who do not know left from right. Add in the

illustration caption: Children do not always know the meaning of or even hear the words we use to give direction.)

Teacher: Plus one. Please write the number for what you have on your trading board on your recording strip.

(illustration 12-4-18)
(Show a trading board with one square on it. Then show the different ways the students record it. One example, a child who has written two, because he (or she) counted the drawn square as a square on the board. Show students who have recorded the one on the wrong (left) side of their three column recording. Show recordings like 000 and just 1 with no 00. Show the recording paper from *Mathematics... a Way of Thinking* (three separate recording paper sections on the same page, not cut apart) with the one written correctly in terms of the actual recording strip, but in three different places on the paper. Some students start at the left strip, some start at the right and some start in the middle.)

Teacher: Continue doing plus ones on your own and record the numbers on your recording paper.

(illustration 12-4-19)
(Show at least three recording strips. The first two ways are illustrated below. The third way is from the student who simply leaves out numbers when recording.)

000	**000**
001	**001**
002	**002**
003	**003**
010	**010**
011	**020**
012	**030**
013	**100**
020	**200**
021	**300**
022	
023	
030	

(Include in illustration caption: Some students record plus one according to the rule for bases. Others do not understand the rules or abandon the materials too soon.)

At every stage of this lesson the teacher was besieged with requests for help. The frustrated teacher wondered why this way of teaching was supposed to bring more understanding when no one seemed to understand.

At the end of class, the teacher thought about her plus-one lesson. She wondered why her students had not done what she had planned for them to do. She wondered what she could do to make the lesson work. The next time she taught the lesson, she:

Provided an example of a trading board on the overhead.
Talked about the number of folds necessary to get three columns on a paper.
Defined left and right for her students and had them mark left and right on their paper.
Drew a square, cup and bowl on her overhead trading board.
Had neighbors checking neighbors at each step.
Cut the recording strips apart in advance of the lesson.
Provided masking tape for joining the cut strips together.
Recorded plus ones on an overhead recording paper to accompany the students' recordings.
Did not turn the students loose for recording independently until everyone had reached 111.
Asked her students to look for patterns in the square, cup and bowl columns that could help them see what numbers might come next.

The teacher had within herself the ability to ponder the elements of the lesson she presented. She turned her frustration into a search for what went wrong so that she could make it right. The advantage the experienced teacher has over the beginning teacher is the mistakes already made and learned from.

Lesson Five

Purpose	Learn to see place-value patterns within and between bases. Learn to use materials to prove answers.
Summary	Students record values for different bases on a single matrix and examine the matrix for patterns. Then students use the multibase blocks (if available) to create, solve, and then check problems.
Materials	Squares, cups, bowls, big bowls, blank matrix, calculators, multibase blocks (if available).
Topic	Looking for patterns in 1000, 100, 10, 1.
Topic	Addition problems using multibase blocks.
Topic	Subtraction problems using multibase blocks.

Blocks, not chips...

First teacher: (Sitting at a table in the Teachers' Room) How many knives do I have here?

(illustration 12-5-1)
(Two knives lying side by side.)

Second teacher: Two.
First teacher: No. I have eleven. These knives are chip-trading knives.

Squares, cups and bowls give our students a framework for understanding how numbers work. In some cases, we add a fourth column to our trading boards so that students may experience even larger numbers. But there is a limit to the kinds of problems we can solve with squares, cups and bowls.

In *Mathematics... a Way of Thinking,* the lessons with cups and bowls are followed by lessons with trading chips.

> If students are to be able to deal with increasingly larger numbers, representing increasingly abstract notions of quantity, they need to internalize or abstract what they have learned about grouping and regrouping. The chips...provide the necessary link between the concrete representation of [squares] and cups and the more abstract concept of numbers representing unseen quantities. (*Mathematics... a Way of Thinking,* page 101)

The purpose of the chips in *Mathematics... a Way of Thinking* is to provide students with an aid to computation.

> In Japan, students are allowed to work calculations on their sorobons. Japanese students register the highest level of arithmetic understanding of any of the nations tested by UNESCO, including the United States. The chip trading board is the equivalent of a...sorobon or abacus. (*Mathematics... a Way of Thinking*, page 113)

In *Mathematics... a Way of Thinking*, which was written before calculators were commonly available, chip trading was used to give students a multibase abacus. Now, our students have ready access to calculators. They no longer need an abacus or a set of chips to find the answers to the larger addition and subtraction problems in class.

Squares, cups and bowls are concrete manipulative materials. They provide a physical model for our students of what numbers mean. In any base, the number 10 means there is one cup on our trading board and no squares left over. The base we are working in tells us the number of squares in the cup. There is no abstraction here at all.

When chip trading is introduced, colored pieces of paper are used to represent the cups, bowls and bigger bowls. Chips are exchanged for chips in the same way that we exchange five one-dollar bills for a single five-dollar bill. Students may manipulate the chips, but that does not make the chips concrete. The chips are as abstract as the coins we use. Five pennies make a nickel, but we cannot see or feel the pennies inside any nickel we might hold.

(illustration 12-5-2)
(111 (base 4). Write it. Show it with chips on a chip trading board. Show it with a bowl, cup and square on a square trading board. The bowl and cup are filled with the appropriate numbers of squares.
Caption: How many squares can the student see? How many chips?)

Multibase blocks are a material to use between lessons with squares and lessons that use calculators.

(illustration 12-5-3)
(Multibase blocks, displayed by base, with each base labeled. Include a caption that says, "Multibase Blocks".)

In the best of all possible worlds, all of our classrooms would have multibase blocks. Each different piece represents concretely the smaller pieces from which it is derived. Once students understand the quantities that the pieces represent, they can more easily manage the exchanges. Multibase blocks end the endless counting of squares into cups and bowls.

Lesson Five is a lesson on multibase blocks. If we have the blocks, we should feel pleased with our good fortune. If we do not have the blocks, we use squares, cups, bowls and big bowls instead.

Cubes, longs, flats, blocks...

Teacher: How many flats does it take to make this block?

(illustration 12-5-4)
(A single base-ten block.)

First student: Five.
Second student: No, six! There's one on the bottom, too.

The multibase blocks make so much sense to us that we might assume our students see what we see. But when asked to say how many cubes it takes to make a block of any base, students often count only the cubes they see. They do not always know that blocks have cubes inside. All the blocks in all the bases have six flats for sides. When asked how many flats to make a block, some students may count only the flats they see.

Teacher: Here is a matrix that may help us look for patterns in the bases.

	block	**flat**	**long**	**cube**
base 2				
base 3				
base 4				
...				
base 10				

How many cubes to make a cube in base three?
Students: ?
Teacher: I'll give you a hint. The answer is one.
Teacher How many cubes to make a long?
Students: Three.
Teacher: How many cubes to make a flat?
Students: Nine.
Teacher: How many cubes to make a block?
Student: Fifty-four.
Teacher: How did you get fifty-four?
Student: I counted nine on one side and multiplied by all six sides.
Teacher: That way might work, but I think it counts some of the cubes twice. There are cubes that are on the edge of the block, so when you count all the cubes on one side, you have already counted some of the cubes on the side right next to it.
Let's try a different way to count up all the cubes. What base is this?
Students: Three.
Teacher: How many cubes to make a long?
Students: Three.
Teacher: And what base did you say this is?
Students: Three.
Teacher: How many longs to make a flat?
Students: Three.
Teacher: And what base?
Students: Three.
Teacher: How many flats to make a block?
Students: Three.

Teacher: You just told me that nine cubes were in a flat, so if three flats are in a block, how can you use what you know about cubes in a flat to figure out how many cubes in a block?
Student: Add nine up three times.
Teacher: Do the adding and tell me what you get.
Students: Twenty-seven.

	block	flat	long	cube
base 2				
base 3	**27**	**9**	**3**	**1**
base 4				

Teacher: What I want you to do is to fill in this matrix for all the bases. I want to know how many cubes make one cube, one long, one flat and one block in each of the bases that we have in our set of multibase blocks. And, if you think you can see patterns, you can fill in the numbers for the bases we do not have.
You may work with a partner for filling in your copy of the matrix, and you may use your calculators to help you if you wish.

The students work in teams and share their findings with each other.

Teacher: Tell me what you find, so I may record it on my overhead matrix.

	block	flat	long	cube
base 2	**8**	**4**	**2**	**1**
base 3	**27**	**9**	**3**	**1**
base 4	**64**	**16**	**4**	**1**
base 5	**125**	**25**	**5**	**1**
base 6	**216**	**36**	**6**	**1**
base 7	**343**	**49**	**7**	**1**
base 8	**512**	**64**	**8**	**1**
base 9	**729**	**81**	**9**	**1**
base 10	**1000**	**100**	**10**	**1**

Are there any patterns in the columns on my matrix?

The patterns in the columns for cubes and longs are easy to see. Our students may need help in seeing the patterns for the flats and blocks.

Teacher: You may use your calculators to help you answer these questions if you wish. Let's look at the numbers in the flats column. How much larger do the numbers get as we move down the column? How much larger is nine than four? How much larger is sixteen than nine?

We always ask our students to look for patterns in the numbers they see, but sometimes the patterns are not easy for them to discover. Sometimes the patterns are not even easy for us to see! To help students see the patterns already there, we teach a technique that helps to reveal them.

Teacher: Sometimes when I don't think I see a pattern, I look for the differences in the numbers to see if there might be a pattern. Let's see what the differences are between all the numbers in the flats column.

flat	difference
4	
	5
9	
	7
16	
	9
25	
	11
36	
	13
49	
	15
64	
	17

81	
	19
100	

Teacher: Can you see a pattern for the differences?
Student: The number gets bigger by two each time.
Teacher: Okay. Lets look for a pattern in the blocks. Sometimes it takes finding the differences in the differences to see the pattern that is there. Let's look at the differences for the blocks. Can you see the pattern yet? If not, then take the difference of the differences again. Can you see the pattern yet? Let's take the differences one more time to make sure we do. Can you see it now?

8			
	19		
27		**18**	
	37		**6**
64		**24**	
	61		**6**
125		**30**	
	91		**6**
216		**36**	
	127		**6**
343		**42**	
	169		**6**
512		**48**	
	217		**6**
729		**54**	
	271		
1000			

Calculators make it much easier for our students to explore differences.

Teacher: Let's see what other patterns we can find. Let's look at the row for base three.

	block	flat	long	cube
base 3	**27**	**9**	**3**	**1**

Look across the numbers. How many ones to get three?
Students: Three.
Teacher: Three times one is three. What base is this?
Students: Three.
Teacher: How many threes to get nine? You may use your calculator to help you figure this out.
Student: Three plus three plus three.
Teacher: How many threes is that?
Students: Three.
Teacher: Three times three is nine. What base is this?
Students: Three.
Teacher: How many nines to get twenty-seven? Use your calculator if you wish.
Student: Nine plus nine plus nine.
Teacher: How many nines is that?
Students: Three.
Teacher: Three times nine is twenty-seven. What base is this?
Students: Three.

There is a pattern to the questions we ask:

Teacher: Let's look at the numbers for base four. Do you think we will find patterns there?

	block	flat	long	cube
base 4	**64**	**16**	**4**	**1**

We repeat the base three questions for base four. And then we look at base five, base six and so on. We study the matrix on the overhead for all the patterns we can find.

The emphasis is different...

When our students are familiar with the cubes, longs, flats and blocks, they create and record addition and subtraction problems in the same ways they created problems for squares, cups and bowls. The techniques are the same, but the emphasis is different.

(illustration 12-5-5)
(Show the steps of an addition problem worked out with multibase blocks. Three dice rolled and set out in the top spaces of the flats, longs and cubes columns. Flats, longs, cubes in top row to replace the dice. Three dice rolled and set out in the second row of spaces for the flats, longs and cubes columns. Flats, longs, cubes in the second row to replace the dice. The problem recorded in numbers on paper. Show the completed addition, with the materials moved to the bottom row and all the exchanges made. Show the finished problem written on paper.)

As our students recorded problems for squares, the emphasis was on letting the answer flow from the materials. Once students understand the concepts of adding and subtracting, the focus shifts to using materials to prove the answers.

Teacher: Today as you make up problems, I would like you to think about what the answer to each problem might be before you work the problem with your multibase blocks. Write your answer on your paper and then use your materials to see if your answer is correct.

Calculators give us answers, but how do we know the answers are correct? Materials are what we use to prove our work. We ask our students to anticipate in their minds what the materials will do, then use the materials to see if they are right.

As our students begin anticipating the answers the materials will give, we show them techniques of keeping track of what they know.

Teacher: When I think about the answers I might get when I use the materials, I keep track of what I am thinking by writing notes. I'll show you what I mean.

$$\begin{array}{r} \mathbf{124} \\ \underline{\mathbf{234}} \end{array} \quad \textbf{base 5}$$

Let's start with the cubes column. What do you think four cubes plus four cubes will be in base five? Will I just get cubes or will there be any longs, with cubes left over?
Student: You will get one long and three cubes.
Teacher: That's what I think will happen too, but I am going to write down what I think will happen for the whole problem before I check my predictions with the blocks. It helps me to keep track by writing down the cubes and longs.

$$\begin{array}{r} 1 \\ \mathbf{124} \\ \underline{\mathbf{234}} \\ \mathbf{3} \end{array} \quad \textbf{base 5}$$

Why do you think I wrote a one above the longs column?
Student: To remember you put one long there.
Teacher: What do you think we will get when we add one long plus two longs plus three longs?
Student: Six longs.
Student: You can't have any sixes in base five. You get one flat and one long left over.

$$\begin{array}{r} 1\,1 \\ \mathbf{124} \\ \underline{\mathbf{234}} \\ \mathbf{13} \end{array} \quad \textbf{base 5}$$

Teacher: Why did I write one down beneath the longs column and one above the flats?
Student: Because you had one long left over and one flat to add to the other flats.
Teacher: What do I do now?
Student: Add the flats.
Teacher: What do you think we will get when we add one flat plus one flat plus two flats?
Students: Four flats.

Teacher: Do we have enough flats to make a block?
Students: No.

```
  1 1
  124
  234
  ---
  413      base 5
```

Teacher: Now let's do it with the materials to see if we are right.

Subtraction notes...

For subtraction, students start with a block in the top row. Students roll the dice once to find numbers for the flats, longs and cubes to share the top row with the block and then roll again to know the quantities to subtract.

As our students use materials to find answers to their problems, we have them keep a visual record of the steps along the way.

(illustration 12-5-6)
(Show the steps of a subtraction problem worked out with multibase blocks. Block alone in top row. Three dice rolled and set out in the top spaces of the flats, longs and cubes columns. Block, flats, longs, cubes in top row to replace the dice. Three dice set in the flats, longs and cubes columns in the bottom spaces on the trading board. The problem recorded in numbers on paper. Show the completed subtraction, with the materials that were taken away placed beneath the dice in the problem. Show the finished problem written on paper.)

When our students keep the flats, longs and cubes they subtract directly beneath the dice, they can prove to us and to themselves that they have taken away exactly what the dice have told them to. All the materials are accounted for.

Teacher: Here's a way that some people say works for checking answers to subtraction problems. Start with the same top row of numbers again, and subtract the answer that you just got. The answer that you get now should be the numbers that you subtracted the first time.

```
 1023                    1023
 -134                    -334
 ----                    ----
  334   Base 5
```

Why do you think this method works? Would this method work for any base?

Lesson Six

Purpose	Learn the names for larger numbers.
Summary	Students learn to read large numbers.
Materials	None.
Topic	Learn to read large numbers.

Names for the numbers...

As our students work in different bases, we teach them how to read the numbers. The number 10 in any base is read "one, zero." The number 14 in base five is read "one, four, base five." The number 232 in base four is read "two, three, two, base four." As our students first begin work in base ten, we have them read the base-ten numbers in the same way that they read numbers for any other base: 10 is "one, zero, base ten;" 14 is "one, four, base ten;" 232 is "two, three, two, base ten."

We ask our students to read the base-ten numbers the same way they do any other base to make the connection between other bases and base ten. Once the connection has been made, we allow our students to use the names for numbers in base ten that they already know. 10 is ten. 14 is fourteen. 232 is two hundred thirty-two.

Allowing students to use the base-ten names for numbers does not guarantee every child knows the names for all the numbers in the problems they create. Students can find right answers with materials and not know how to say the numbers in the answer they have found. When we teach our students to find answers, we also teach them how to read the answers that they find.

The number 234 means two bowls, three cups and four squares. We read that number as "two hundred thirty-four." The number 204 means two bowls, no cups and four squares. We read that number as "two hundred four." Our students learn not to say a number for an empty space. Our students read numbers to us or they read numbers to each other.

Once students can read any number from 0 to 999, we teach them how to read much bigger numbers. To read a larger number, students need only know how to read the comma in between. The first comma means thousand. All the other numbers read the same:

432 is four hundred thirty-two.
432,432 is four hundred thirty-two thousand four hundred thirty-two.

When our students can read the first comma as a thousand, we teach them that the second comma means a million:

543 is five hundred forty-three.
543,543,543 is five hundred forty-three million five hundred forty-three thousand five hundred forty-three.

The commas after million are:

Billion	1,000,000,000
Trillion	1,000,000,000,000
Quadrillion	1,000,000,000,000,000
Quintillion	1,000,000,000,000,000,000
Sextillion	1,000,000,000,000,000,000,000
Septillion	1,000,000,000,000,000,000,000,000
Octillion	1,000,000,000,000,000,000,000,000,000
Nonillion	1,000,000,000,000,000,000,000,000,000,000
Decillion	1,000,000,000,000,000,000,000,000,000,000,000

We can read the biggest numbers as long as we know the name for every comma that the number has.

Lesson Seven

Purpose	Learn that addition and subtraction are tools for finding out. Learn to apply math skills to problems in real life.
Summary	We work with our students to find and then solve real problems that use the skills that our students possess. Students keep a written record of their work.
Materials	Paper.
Topic	Problems drawn from stories.
Topic	Problems drawn from questions.
Topic	Problems drawn from life.
Homework	We send a list of questions home for parents to contemplate with their children.

From the obvious to the obscure...

As our students learn addition and subtraction, we give them opportunities to use their skills. The skills we teach help our students solve problems in their lives. Lessons in mathematics are lessons in life.

There are so many ways to make mathematics meaningful to our students, where do we start? We start with the problems that are already there. We begin with the obvious and move to the obscure.

Stories...

School is filled with stories. Stories are filled with opportunities for doing math.

Sometimes the numbers are already there.

St. Ives
As I was going to St. Ives,
I met a man with seven wives.
Each wife had seven sacks,

Each sack had seven cats,
Each cat had seven kits.
Kits, cats, sacks and wives,
How many were there going to St. Ives?

Questions we might ask:

How many men were met on the way to St. Ives?
How many wives?
How many sacks?
How many cats?
How many kittens?
Have we seen these numbers before? [Look at the matrix in Lesson Five (page 312).]
How many men, wives, cats and kittens altogether?

There Was an Old Woman

There was an old woman who lived in a shoe.
She had so many children she didn't know what to do.
She gave them some broth without any bread,
And whipped them all soundly and sent them to bed.

Questions we might ask:

How many children do you think the old woman has?
How old do you think the oldest one is?
How many years between the youngest and the oldest?
How many years between each one?
What would it cost her to take them all to McDonald's or to the movies?
What would it cost to buy them all shoes?
Why do you think the old woman doesn't know what to do?

There are questions we might ask for any and all kinds of books, even before we begin to read them.

When was this book written? What is its copyright date?
How many years ago was that?
Was this book already written when your parents were in school?
How old do you think the author or illustrator was when the book was written?
How old is the author or illustrator now?
If we are ordering Scholastic Books for our class, how many dollars will we need to send in?
How many dollars would it cost to buy one each of all the books on the Scholastic Book list?
How much did it cost to put all the books in our school library?

As we read any book, what can we think of to ask? What can our students think of to ask?

The book the teacher is asking about today is: *John Patrick Norman McHennessy - The Boy Who Was Always Late*, John Burningham. New York: Crown Publishers, Inc., 1987.

Teacher: What is the name of the book we just finished reading today?
Students: John Patrick Norman McHennessy.
Teacher: What math or number questions do you think we could ask about the story? Don't worry about the answers yet. Let's just think about the questions we might ask.
Student: How many sentences did John have to write on the board?
Teacher: Okay. That is a question we could ask. Don't worry about figuring out the answer yet. Think of another question we might ask.
Student: What time was it in the morning when John set off on the road to learn?
Teacher: Okay. We could ask that question. Another question?
Student: Where were all the other children in his school?
Teacher: You mean in his class or in the whole school?
Student: In his class.
Teacher: Can you make your question into a mathematics or a number question?
Student: How many children were there in John's class?
Teacher: Okay. We could ask that question, too. How about a geometry question?
Student: What were the shapes of the things that John was carrying to school?
Teacher: Okay. Another mathematical or numerical question?
Student: How old was he?
Teacher: Okay. Another question?

As the students ask, the teacher writes the questions down. If a question occurs to the teacher that no student has thought to ask, the teacher can do the asking and add the question to the list.

Teacher: How much do you think John's parents had spent on all of the different items that John lost on the way to school? How far away do you think his home was from his school?

The lesson we are teaching involves addition and subtraction. The questions our students ask may involve so much more. How do we keep the problems focused on just the skills that we are teaching? Should we even try?

We can ask "How many sentences on the board?" because the question fits the adding lesson. But the focus of the problems we create is on the thinking, and not on any one specific skill. As we look for ways to use addition and subtraction, the focus of the problems we select is on the reasoning and the understanding they require. The calculator has all the adding and subtracting skills we need. There is no need to limit the problems we select to the skills that we have taught. The focus of the problems is on the solving. There is no more useful skill that we could teach.

How many pencils?...

Teacher: I am going to give you a problem to solve. I don't want you to tell me what you think the answer might be right away. I want you and your partner to talk to each other and decide how you might go about finding the answer.

How many pencils are in our class? Talk with your partner and think about how you might find the answer.

The students work in pairs.

Teacher: Raise your hand if you and your partner think you know a way to find the answer to my question. Please be sure to listen to the ways your classmates describe their answers, so we can decide together a good way to solve the problem.

Kyle: Nine hundred and thirty-five.

Teacher: Think of what my question is. I want to know how to go about finding out the number. Does your answer tell me a way that I might use?

Andrew: Well, there are twenty-eight people in class, and we estimated that each person has about ten pencils, so...

Teacher: I want to know the exact number of pencils in the room. No estimates, please.

Angela: We think there are one hundred and forty-three pencils.

Teacher: What did my question ask you to think about?

Angela: How many pencils?

Teacher: I want you to tell me how you think you would find the answer. I'm not ready to hear an answer yet. Arthur?

Arthur: Well, there are twenty-eight people in class, and everybody has two pencils each, so that would be fifty-six pencils.

Teacher: How many people in class have more than two pencils? Please raise your hands. I see that almost everybody has more than two pencils. Arthur, how many pencils do you have?

Arthur: Five.

Teacher: Why did you decide to say everybody had two pencils when you have more than two?

Arthur: (No answer)

Brett: Everybody rips open everybody's desk and grabs all their pencils and counts them.

Teacher: I would prefer a less violent solution, thank you.

Danielle: Everybody counts their pencils and then everybody walks around to everybody's desk and asks them how many pencils they have.

Teacher: If everybody is walking around to everybody's desk, then there would not be anybody still at their desk to answer anybody else's questions.

Aaron: Every group counts its own pencils. Then the secretary from each group meets together with the secretary from all the other groups and they add all their numbers together.

The teacher in this class had previously formed work groups. Each work group had a person designated as secretary who would collect and pass out paper and supplies to the group.

Teacher: Would that way work?

Monica: But who would count the teacher's pencils?

Kyle: The teacher counts her own pencils and then she tells her number to the secretaries.

Teacher: Do you think Aaron's suggestion is a workable one? Write your vote, yes or no, on your chalkboard and hold it up.

Students: (Majority of yes answers on the boards)

Teacher: Let's try Aaron's suggestion. Where shall the secretaries gather to share their numbers?
Pat: By the overhead projector.
Teacher: Okay. Once your group has finished counting its pencils, send your group's secretary to the front of the room by the overhead with your total. When the rest of your group is done with the counting, you may free explore your Power Blocks while the secretaries find the answer to my question.
Ryan: Do we count colored pencils too?
Teacher: Should we count colored pencils?
Students: Yes!
Teacher: Then count the colored pencils for your totals.

Seven working groups count their pencils. Each group dispatches its secretary to meet with the secretaries from every other group.

Teacher: What numbers do you have?
Seven secretaries and the teacher say:

64, 71, 57, 76, 84, 101, 63, 15.

Teacher: How can we combine all these numbers? If we use your calculators to put them all together, how will we remember the numbers that we started with?
Jesse: We can each write our number on the overhead. Then we can add the numbers on the overhead to see what the total is.
Teacher: Let's try it.

(illustration 12-7-1)
(The numbers written in a column on the overhead with the total 531 written underneath. The word *pencils* is written by the total.)

Problem solving starts with knowing what the problem is. The next step is planning out a way to solve the problem. Only when we know the problem and the plan for its solution do we gather the numbers that we need.

Kyle and Angela gave 935 and 143 as answers to the teacher's question. They gave numbers for their answers, because numbers were the only answers in mathematics they had ever had to give before. Kyle did not understand the question. Angela did not pay attention to the teacher's answer for the number Kyle gave.

Andrew and Arthur made estimates. It did not occur to them that they could think of ways to solve a problem that would involve gathering actual data from the members of their class. They could only think of ways that involved guessing. Guessing is a way of solving problems they had used before. Arthur proposed an estimate of two pencils for each child, even though he had more than two.

Brett would tear everything apart, but Brett did not expect anyone to take his proposal seriously. He did not relate the solution of the problem to something he could really do. What kind of a question was "How many pencils?" anyway? To Brett, math is putting numbers down on paper. What could finding the total number of pencils in class have to do with math?

Danielle was beginning to get the idea of what a problem-solving plan might be. She heard the students before her call out 935 or 143 or estimation strategies. As she heard each new answer find rejection in its turn, she reasoned that finding the number of pencils must take counting of some kind. Is not counting what she used outside of school to find how many there were of anything? But Danielle had so little practice with the realities of finding number answers that she could not yet think of how to ask everybody to ask everybody when everybody would be asking everybody else.

Was Aaron randomly the wisest child of them all? He did what any child would do when there are other children from whom to learn. He learned his problem-solving strategy from the failed strategies of the children who had gone before. Aaron's plan kept changing as he heard all the other children's plans the teacher said were not acceptable. We may wonder what Aaron's suggestion would have been if he had been the first child called and not the seventh.

The other students in class had no difficulty seeing that Aaron's plan would work. They, too, had listened to the suggestions from the first six. Any child in seventh place that day may have had as wise a plan. Kyle first gave 935 as the solution because he did not understand the question. But it was also Kyle who proposed to have the teacher's pencils counted in. Kyle had learned by listening to the other suggestions made in class just as Aaron had. We all learn by listening to one another as we model with the whole class what it means to problem solve.

Of all the questions we might ask...

Of all the questions we might ask each day we need only ask a few. The number of problems students solve is not the key to learning. What matters is the quality of the question that we pose.

Teacher: Does the money earned by the cafeteria pay for all the food?

What are all the questions the students will have to ask and answer before they can answer just this one? How do they go about gathering all the data they will need?

Student: Let's just ask the cafeteria people if they make enough.
Teacher: We can do that. But we can also help them to find out. What kinds of questions would the cafeteria people have to answer for themselves before answering our questions?

If our students do not know the questions they should ask, we can help by providing suggestions, or we can have a cafeteria worker teach us the kinds of questions that they ask.

How many children eat each day?
Is every day the same?
What is the cost of all the food?
What is the cost of all the cooks?
What happens to the leftover food? Do we use it or do we throw it out?
What other costs do they have to pay? Electricity. Cleaning liquids. Water bills. Straws. Napkins. What else?
What did all the sinks and ovens and cooking utensils cost? Do we need to know these costs to know the price of every meal?

How much more will our students appreciate the cafeteria in our school when they know the effort spent on their behalf? How much better will the cafeteria workers feel knowing some students care to ask?

Menus...

Menus are a natural lesson in adding and subtracting. We gather single copies of the menus from local fast-food or sit-down restaurants. With permission, we make copies for everyone in class. We ask our students to perform the arithmetic involved.

How much will a meal cost us?
How much money should we bring?
What change may we expect?
Which is the cheapest place for us to eat?
Is it cheaper at the restaurant or at home?
Where is the food most expensive?
Why do some places cost so much more?
You have four dollars (or twenty-four, depending on the restaurant) to spend for your dinner. Plan your meal. No, you do not have to include the tax. Not yet.

We give the parents of our students a list of the kinds of questions they might ask when dining out with their children. The children can take paper, pencil and a calculator to the restaurant to practice what they know. Parents can bring menus home for ordering in advance.

Sports...

According to a *Sports Illustrated for Kids* poll, 93% of children ages eight through twelve participate in at least one organized sport.

One teacher in a midwestern town where basketball was king had each student in her class pick a player from the local professional team to follow for the season. What kinds of math questions could the teacher ask about the players or the team after every game?

How many points did the player score last night?
Was that more or less than for the game before?
How many points so far for the season?
How many points in the game for the team?
How many baskets were tried?
Which team had the most basket tries?

Which team had the most points?
How many 1, 2, or 3 point baskets were scored by your player?
What kinds of baskets for the team?
How many fouls did your player commit?
How many fouls for the whole team?
Did the other team have more or less points?
How many minutes did your player play last night?
How many minutes did all the players play?
Which players played the most?
Which players played the least?
How many minutes are in a game?
If there are five players on the team and each player plays every minute of the game, how many minutes altogether did all five players play?
Does the total of all the players' different playing times equal the total playing minutes in the game?

Depending on the mathematical skills we have introduced in class, other questions the teacher could ask as the season progressed are:

What is a shooting percentage?
What percentage does your player shoot?
What other percentages are a part of every game?
How many miles do the players travel to each away game?
How long does it take the team to get there?
Where does each new win or loss put the team in the standings?
How tall is the player you are following?
How much taller is your player than you?
How much does your player weigh?
How much heavier is your player than you?
How high up is the basket?
How big is the basketball court?

What is the math in the other sports our students watch or play?

Shopping...

Two parents on a week's vacation with their elementary-school children gave each child twenty dollars to spend in a souvenir shop on the last day of the trip. The children were given a pad of paper and a pencil and told to do all the arithmetic involved. The twenty dollars did not include the tax. Planning what to buy ended up taking each child about an hour.

The children spent the hour making an ever-changing pile of all the different favorite items they might buy. The price of every item had to be added to all the items already on the list. Each new total had to be subtracted from the twenty-dollar amount so the child could know what was left to spend. When the twenty-dollar amount was exceeded by the total of the items picked, more expensive items were replaced with less expensive ones. So many choices to be made. So much adding and subtracting to do. Not every family can afford twenty dollars, but every family spends money. Shopping in any store is an arithmetic lesson waiting to begin.

In school we can give each team of students a catalog from our junk mail and say:

You have a hundred dollars to spend.
You cannot buy more than one of anything.
Work with your partner to plan what you would buy.
Be prepared to tell us all what you bought, the total cost and your change.

Home...

We can guide our students' parents in teaching mathematics at home by giving them a list of questions they might ask:

How many minutes will each dish in the meal take to cook?
When should we put each dish in the oven or the microwave to have the whole meal ready by six?
How much does it cost to feed the family for a week or month?
How much do all the clothes for all the children cost?
How many sandwiches can we get from that loaf?
Is the phone bill we received added right?
How does the electric meter get read?
How much does it cost to drive the family car?

How much does it cost to go to the movies and to eat the popcorn there?
Is going to the bowling alley or to the ice skating rink a better buy?
How many hours each day is the family television on?
How many hours are spent sleeping in a week or month?
Do we spend more time in a week playing, sleeping, or watching TV?
How many ads are in each half hour of a TV show?
Do children's or adults' TV shows have more ads?

Our list of questions helps parents remember that learning is a natural part of life. Learning is not something that needs a page of problems from school before it can take place.

Problem-solving skills...

We model thinking and wondering for our students by doing our own thinking and asking our own questions. Sometimes we ask good questions, and sometimes we ask questions that are not so great—and we do not always know which questions are which. Some questions will have answers our students can find. Some questions may need more skill before an answer is discovered. Some answers may have to wait until we are all much older. There is no certainty that every problem we pose will be a problem our students can solve. Wondering does not guarantee knowing.

Ideally, we would not ask our students to find answers until we have given them all the necessary answer-finding skills. But the questions we or our students ask may pose problems that are better solved with skills we have not yet introduced. What do we do when our questions go beyond taught skills? We do the best we can.

When we ask a question, our students think of what the question means, and they discuss the problem it poses with their partners. To find an answer they can prove, they use the resources in the room: calculators, cubes, blocks, pencils, paper, scissors, rulers, crayons, each other, whatever is around. If they do not know how to work the problem or even what the problem is, we use our best Socratic logic to guide them in thinking about what they know. If students do not know what to do, they can tell us what they might like to try. Not knowing and trying anyway is safe to do in our class.

Working together, sharing and writing are all a part of math. Students who do not know what to write can tell their answer to a friend who can record the answer on paper. Writing is just showing on paper what we say with words. Saying and writing are the same.

None of us is as smart as all of us. When we give our whole class problems to solve, our whole class amazes us with how good at problem solving it can be.

Assessment...

Assessment is not a separate activity. If we want to know if a student understands addition and subtraction, one of many ways we find out is to give the student several menus and ask:

How much will a meal cost us?
How much money should we bring?
What change may we expect?
Which is the cheapest place for us to eat?
Is it cheaper at the restaurant or at home?
Where is the food the most expensive?
Why do some places cost so much more?
You have four dollars (or twenty-four, depending on the restaurant) to spend for your dinner. Plan your meal. No, you do not have to include the tax. Not yet.

Do the questions look familiar? We asked them once before. Assessment is what we have been doing all along. Assessment means finding out if our students can use what they have learned. We ask our students to prove their answers and write their proofs to share. We place selected written proofs in each student's portfolio as evidence of what each student knows.

Lesson Eight

Purpose	Learn to look for patterns in numbers everywhere.
Summary	Students look for patterns between bases, in palindromes and in ordinary events.
Materials	Palindrome blackline. Plus-one strips from Lesson One.
Topic	Compare plus one strips from different bases.
Topic	Palindromes.

Topic	Examine situations for patterns.
Homework	Pattern situations are sought at home.

Looking for relationships...

Not all of learning's value can be measured by its application to our lives. Some numbers that we ask our students to explore are not used to make us better shoppers or more aware of the time until lunch.

Teacher: Let's look at the recording strips we made for plus one in several different bases. Let's see what patterns can we find.

(illustration 12-8-1)
(The plus one recording strips for several bases. Show the numbers from 0 to 100.)

base 2	base 3	base 4	base 5	base 6
000	000	000	000	000
001	001	001	001	001
010	002	002	002	002
011	010	003	003	003
100	011	010	004	004
	012	011	010	005
	020	012	011	010
	021	013	012	011
	022	020	013	012
	100	021	014	013
		022	020	014
		023	021	015
		030	022	020
		031	023	021
		032	024	022
		033	030	023
		100	031	024
			032	025
			.	.
			.	.
			.	.

Teacher: How many different lines did you use to record numbers on your base-two recording strip from zero, zero, zero to one, zero, zero?
Students: Five.
Teacher: How many different lines did you use to record numbers on your base-three recording strip from zero, zero, zero to one, zero, zero?
Students: Ten.
Teacher: How many different lines did you use to record numbers on your base-four recording strip from zero, zero, zero to one, zero, zero?
Students: Seventeen.
Teacher: Is there a pattern to the numbers?
Students: ?
Teacher: Let's find the differences between the numbers and see what we can see.

numbers	**difference**
5	
	5
10	
	7
17	

What do you think the next difference might be?
Students: Nine.
Teacher: Then what do you think the number of lines will be for base five?
Student: Seventeen plus nine.
Student: Twenty-six.
Teacher: Count the number of lines between zero, zero, zero and one, zero, zero in base five and see.

numbers	difference
5	
	5
10	
	7
17	
	9
26	

And what do you think the next difference might be?
Students: Eleven.
Teacher: Why do you think it will be eleven?

What other patterns might our students see as they compare plus one in different bases?

Palindromes...

Teacher: Today we are going to look at palindromes. A palindrome is a word or a number that reads the same frontward or backward, like "Anna" or "a Toyota" or like 121 or 42324.
I am going to take the number 12, reverse it, then add the 12 to the reversed number, like this:

$$\begin{array}{r} 12 \\ \underline{+21} \\ 33 \end{array}$$

What number did I get?
Students: 33.
Teacher: Is 33 a palindrome?
Students: Yes.
Teacher: Now, I'll try 56. What is 56 reversed?
Students: 65.
Teacher: Okay. Then we add the two numbers together.

$$\begin{array}{r} 56 \\ \underline{+65} \\ 121 \end{array}$$

What number did we get?
Students: 121.
Teacher: What am I going to ask you about this number?
Students: Is it a palindrome?
Teacher: Is it?
Students: Yes.
Teacher: Let's try 75. What do we do first?
Student: Reverse it and add to get the palindrome.
Teacher: That's what we do first and second. Let's do both and see if we get a palindrome.

$$\begin{array}{r} 75 \\ \underline{+57} \\ 132 \end{array}$$

Is 132 a palindrome? Does it read exactly the same both ways?
Students: No.
Teacher: Then, let's reverse the 132 and add it to itself.

$$\begin{array}{r} 75 \\ \underline{+57} \\ 132 \\ \underline{+231} \\ 363 \end{array}$$

Is the new answer a palindrome?
Students: Yes.
Teacher: We call 363 a two-step palindrome, because we had to reverse and add the numbers two times to get the palindrome. Look at the palindrome recording sheet on the overhead. We did 12 already, so I'll write that one on the recording sheet.

number	steps	palindrome
10		
11		
12	1	33
13		

Let's do 10. What do we do first?
Student: Reverse it and add.

$$\begin{array}{r} 10 \\ \underline{+01} \\ 11 \end{array}$$

Teacher: Do we have a palindrome for an answer?
Students: Yes.
Teacher: How many steps?
Students: One.

number	steps	palindrome
10	1	11
11		
12	1	33
13		

Teacher: Let's try 11.
Student: But 11 is already a palindrome.
Teacher: True. How many steps?
Student: No steps. It already is one.
Teacher: For this particular recording sheet I want you to try every number for at least one step. Try 11 for one step. What would we do?
Students: Reverse the numbers and add.

$$\begin{array}{r} 11 \\ \underline{+11} \\ 22 \end{array}$$

Teacher: Even when we reversed it we still got a palindrome.

number	steps	palindrome
10	1	11
11	1	22
12	1	33
13		

I think I can already see some patterns. How many steps do you think it might take to make a palindrome from 13.
Students: One.
Teacher: And what do you think the palindrome might be?
Students: 44.
Teacher: Let's do it and see.

$$\begin{array}{r} 13 \\ \underline{+31} \\ 44 \end{array}$$

I'll add that to the recording sheet.

number	steps	palindrome
10	1	11
11	1	22
12	1	33
13	1	44

14		
15		

I can already see what you might predict for 14 or 15. Let's try 19 and see what happens.

$$\begin{array}{r} \mathbf{19} \\ \underline{\mathbf{+91}} \\ \mathbf{110} \\ \underline{\mathbf{+011}} \\ \mathbf{121} \end{array}$$

How many steps?
Students: Two.

number	**steps**	**palindrome**
10	**1**	**11**
11	**1**	**22**
12	**1**	**33**
13	**1**	**44**
...	**...**	**...**
19	**2**	**121**

Teacher: That's interesting. When we did 56 we got 121 as the palindrome in one step.
What I want you and your partner to do now is to take a palindrome recording sheet and fill in all the palindromes and all the steps for the numbers from 10 through 99. You may do the numbers in any order you wish. If you and your partner have any difficulty finding a palindrome for any of the numbers, it is okay for you to ask anyone else in class if they have found a number they can share with you. As you work, see how many different palindromes you can predict from all the patterns that you find.

The palindrome recording sheet invites our students to search for patterns. The more patterns they see, the less adding they will need to do. No matter how many patterns our students may see, the recording sheet also involves addition problems larger than any problem on any worksheet we have ever handed out.

Once we show our students how to reverse the numbers and add until the palindrome is found, we watch quietly as our students work. If any team cannot find a palindrome, that team has every other team of students in class as its resource. The students and not the teacher are the resource for any answers that a student cannot find.

I believed...

The palindromes for 13, 14 and 15 are easy to find. Later on, the numbers become much more difficult. Our students may ask us if some numbers have any answer at all. Our reply is that they may check with all the other teams as often as they wish. The only answers are the answers that they find.

When I was first learning about palindromes, a mathematician friend identified for me one particular number for which he said no palindrome could be found, no matter how many times I added and reversed. I believed him. I took his answer as my own.

When I asked my first class of students to fill in the palindrome recording sheet, they came to me to ask if it were possible to find a palindrome for this one number. I believed the number was impossible to do, but I believed as strongly that the answer should not come from me. I did not tell my students what I believed about the number. Instead, I said that any answer was theirs to find.

Most students gave up the search for this particular palindrome, but one student who persevered found an answer. I believed the answer to be wrong. How could this student be right when I knew no answer could be found? I kept my opinion to myself. My students did not need me to prove the answer wrong or right.

Once this student had an answer, other students who had given up started adding to see if they agreed with her or could prove her answer wrong. Students checked each other's work carefully. Were all the numbers reversed correctly? Was all the adding done just right? A second, then a third and then a fourth student found answers that agreed. Then more and more students working with each other proved the answer right. What I believed was right was proved wrong.

Would my belief have been challenged if I had said, "Do not try, I know it can't be done!" Few students have more faith in their own thinking than in any answer that a teacher might provide. If I had told my students that the answer did not exist, they would likely have substituted my thinking for their own, just as I had done when my mathematician friend said it was impossible. We keep our answers to ourselves. When we provide the answers, we eliminate our students' need to think. Mathematics is a way of thinking and we are teaching math.

The underlying sense...

Mathematics is patterns and connections. Numbers show us similarities in seemingly unrelated events. Pattern searches help us learn to look for the underlying sense in the numbers that we see. When students learn to search for patterns everywhere, they become more persistent mathematicians. They begin to look for patterns in any numbers that they see, and they will not give up their search until the numbers make sense in some way. They know there is an answer in the numbers that they can find. They understand the sense that numbers make.

What other pattern-searching activities might we use? We might ask our students to count the people passing by our classroom window from 8:00 to 8:30 in one day. We can count the people for today and count the passersby tomorrow. We can add the numbers together to find the total number for two days, or we can subtract one number from the other to know how many more or fewer people walked by today than yesterday. This kind of adding and subtracting uses the skills that we learn. We can gather data for more days and see what else the numbers show.

	Week 1	**Week 2**
Monday	**17**	**19**
Tuesday	**25**	**28**
Wednesday	**35**	**34**
Thursday	**23**	**22**
Friday	**15**	**14**

We use our skills at adding and subtracting to answer questions like:

How many people walked by each week?
Which week had the most people?
Which week had the fewest people?
How many more or fewer walked by each week?
Which day had the most people walking by?
Which day had the fewest people passing by?

We use what we know of patterns and connections to answer questions like:

Why did that day have the most?
Why did this day have the least?
What would we predict for the people walking by in week three?

What other situations might we find that have numbers we can ask our students to collect and analyze? Will our students be able to find more elaborate patterns on the 0—99 chart as their knowledge grows? Are they ready to examine Pascal's Triangle for the patterns there? What other situations old and new might we explore as our adding skills improve?

Lesson Nine

Purpose	Learn that creativity and inventiveness are problem-solving tools.
Summary	Our students use their inventiveness to solve problems that we give and share their individual or collective inventiveness with everyone in class.
Materials	None outside our heads.
Topic	Kids in class, kids in school.
Topic	Thinking and mental arithmetic. Reasonableness.
Homework	We ask parents to do their mental arithmetic aloud.

Right inside our heads...

With calculators in hand and materials available for proofs, what else do our students need to know about adding and subtracting? They need to practice doing their arithmetic in the same way that we do.

In our own lives, we have calculators available as often as we wish. We have paper and pencils available on demand. So, how do we solve the arithmetic problems we come across each day? Even those of us who would say we are terrible at math solve almost all the arithmetic problems we encounter right inside our heads.

How long will the gallons we have left in this tank of gas last us on this drive? Can we make it to our destination and then back again with this much gas, or should we fill our tank up first?
The road sign says 45 miles to our destination. How long until we are there? About how many miles have we traveled so far?
The cookbook says this recipe will serve four, but we need to serve only two tonight.
The football score said our team won 24 to 7. What was the margin of victory? Did it beat the spread? How were the points most likely scored?
The video our family wants to watch shows a running time of ninety minutes on its label. When is the latest we can start the movie and still send the youngest child off to bed on time?
Can I get from Susie's dancing lessons to Lisa's soccer practice and still get groceries before six o'clock tonight, or should I stop and get the groceries now?
How many calories in this candy bar and Coke?
Do I have enough money with me now to pay for three dinners at Burger King and then three tickets to the movies, or should I pay a visit to the automatic teller first?
The date on this milk carton says October 1st. How many days away is that? Should I buy one or two cartons to last until then?

Thinking, not yet saying...

Tour guide: The gas station over there is the largest gas station in the world. How many gas pumps do you think there are?

(illustration 12-9-1)
(3-D sketch of the gas station pump arrangement from Saudi Arabia. Do not make all of the pumps visible. Show only the necessary components for the estimation.)

Tourist: (Thinking, not yet saying) There are four pumps in each group and four groups in each section. That's sixteen. There are three sections that I can see from the front. That's three times sixteen. That's too hard. Three times fifteen is easier to do. That's forty-five. I can see there are four rows of sections and all the rows probably have the same number of pumps as the section row in front. That's four times forty-five. Too hard. Four times fifty is close enough. Besides, I already dropped some pumps out when I did fifteen, not sixteen. Four times fifty is two hundred.
(Saying) I would guess about two hundred.
Tour guide: That is a very good guess. There are two hundred and eight.

• • •

Parent: Counting you, fifteen boys and fifteen girls are coming to your party. How many kids do we have to buy food for?
Child: (Thinking, not yet saying) Fifteen plus fifteen. Fifteen is ten and five. Two tens is twenty. Two fives is ten. Twenty and ten is thirty.
(Saying) Thirty.

Using what we know...

How do we learn to do arithmetic in our heads? Most often the skills we use in solving the problems we come across each day are skills we make up for ourselves.

The adult estimating the pumps at the gas station made the problem as easy as possible every step of the way. She did not feel obligated to multiply sixteen times three even though she could see there were sixteen pumps in a section. She used what she knew about numbers to make the problem a workable one. The child adding the boys and the girls for his party also used what he knew about numbers to make his problem easy. He did not add the two fives in the units column and then carry the one to the tens column. He took out the tens and added them first. He made no use of column addition at all. He used what he knew about numbers to make the problem workable.

In Lesson Seven we created opportunities for our students to use the addition and subtraction skills they had learned. We surrounded our students with math in the same way that they are already surrounded with language. We made our students aware of the math that already fills their lives.

When we help our students think about the problems they solve inside their heads, we continue the process of surrounding. We help our students see that the mathematics we invent to find answers in our heads is not a separate kind of math. Using mathematics means applying creatively the skills we learn to the problems that we face. We link all the different ways our students invent to solve their problems to the math we teach in school. Mathematics is a way of thinking and thinking is what we do inside our heads.

Mental arithmetic...

How can we teach our students to do mental arithmetic? Our students already solve problems of all kinds inside their heads. We teach them to:

Be aware of anything they know from anywhere they might have learned it.
Use what they know even if it is not what they were taught in school.
Share with each other the variety of approaches that they use.

Teacher: We are going to do some mental arithmetic today. Mental means inside your head. The rule for this game is that you have to think of a way to get the answer without doing anything outside your head. No materials. No calculators. No counting out loud. Use only your thinking.
How many children are here in class today? Don't tell me your answer out loud yet. I want everybody to have time to think about it.

The problems we assign do not have to be difficult. The object of the lesson is not the answer, it is the thinking involved.

Teacher: Now, please share with your partner how you found your answer.

We ask our students to share with each other first as we walk around the room listening to the variety of ways. There is never enough time in our day at school for everyone to share his or her solution with everyone else in class. When we have each student share with another student first, we guarantee that everyone will have the opportunity to express in words the thinking he or she has done.

Writing is an integral part of math, but we do not have our students write about their thinking now. If they wrote, we could not separate out what they thought inside their heads from what they thought to do as they wrote.

Teacher: Tell me some of the ways you thought of to find an answer.
Student: I counted everyone by counting with my eyes.
Student: I counted the two empty chairs. When we are all here, there are thirty. There were two chairs empty, so that meant there were twenty-eight.
Student: There were four rows with five children in them. I counted by fives four times. That was twenty. Then there were two rows with four. Four plus four is eight. Twenty and eight is twenty-eight.
Student: I counted by twos. Two, four, six, eight, all the way to twenty-eight.
Student: Two rows is ten and two more rows was ten more and eight more was twenty eight.
Student: I started with thirty because there are thirty when we are all here and I counted two backwards for the two people who are not here.

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Teacher: This is a mental arithmetic problem I had to work out inside my head at the store yesterday. If you had been there with me, how would you have helped me do it?
I went to the store yesterday to buy sets of batteries for the tape recorders in our classroom. Each set cost $1.99. I had only $7.00 with me. How many sets could I buy?
I didn't have my calculator with me. I did not have any materials either. Share with your partner how you think I might have figured in my head how many sets of batteries I could buy.

If our students can think of ways on their own, we encourage the sharing of all the different methods. If they cannot think of ways, we can share our thinking with them.

Student: We can't add $1.99 in our heads. That's too hard.
Teacher: Whenever I do problems inside my head, I think of ways to make the numbers easy for myself. $1.99 is hard to add, but what number do you get if you add a penny to $1.99?
Students: $2.00.
Teacher: So, if I add a penny to the cost of the sets of batteries, how much would a set of batteries cost?

Students: $2.00.
Teacher: So, how many sets could I buy with my $7.00?

We share with our students the techniques we use. We talk about the thinking process. We talk about the skills we use to find answers that are close enough. "Close enough" is a different kind of answer than we are taught to find in school. When we learn arithmetic in school, there is always an exact answer to be found. When we use arithmetic in our daily lives, we change the numbers so the addition or subtraction will be easier to do.

When we talk with our students about all the shortcuts we and they invent, we help them see that:

Mathematics is flexible and free.
Finding answers is more inventive than simply memorizing all the rules.
Any way we find to get the answer is still called math.
No one who can think should ever have to feel that he or she is terrible at math.

Rounding is a skill we use all the time when we add or subtract numbers inside of our heads. We do not have to wait until we have taught rounding as a lesson in our class to talk to our students about changing hard numbers into easy ones.

Teacher: Talk with your partner first, then tell me your thinking for how many sets of batteries I could buy.
Student: I made $1.99 into $2.00. Then I took $2.00 from $7.00 three times. So you could buy three sets of batteries and have $1.00 left over.
Student: I made $1.99 into $2.00. I knew that three times two was six and four times two was eight, so you couldn't get four because you only had $7.00. So you could get three.
Student: I made $1.99 into $2.00. Then I kept adding two until I got past seven and then I went back to the number before seven, which was six. Then I knew I added three times, so you could get three sets. But you had $1.00 left over, so you could buy a half a set, too. So you could buy three and a half sets. But the sets did not really cost $2.00, so you would get a penny change for each set. So you got three pennies change, or $1.03 change if the store would not let you buy a half a set.

For any problem we present, we help our students' thinking by the kinds of questions we ask:

How might you figure this problem out?
What do you know about the numbers?
How can we make the problem simpler?
Is this kind of problem like any other kind of problem we know how to do?
Can anybody think of a different way we might use?

Reasonableness again...

In *Beginning Addition and Subtraction* we taught our students that answers should be reasonable (page147). Reasonableness means knowing when an answer makes sense. An answer that is reasonable does not mean the answer has to be exactly right.

Teacher: Here is another problem for you to do inside your heads. As you think about ways to find an answer, remember what we did when we were adding $1.99 for batteries. It is okay to make the numbers easier to add.
Here are prices of four things I bought at the store yesterday. About how much money did I need to have with me to buy them all?

$7.19
1.95
2.75
+2.25

Students add this column of numbers in their heads and share with each other the methods that they use. It may not occur to anyone that 2.75 and 2.25 is 5.00. More likely, they will round every number up (or maybe even down) and add:

$8.00
2.00
3.00
+3.00

As they talk about their methods, we talk about the ways we use to add inside our heads. We can introduce rules for rounding or we can let every number, regardless of its size, be rounded up. We can look only at the dollars or we can talk about ways to add the pennies first. We talk about reasonable ways to find the answers in our heads.

Reasonableness is useful for more than doing problems in our heads. Reasonableness means knowing that the answer we have is an answer that makes sense. Calculators find answers in a flash, but we do not always have materials around to check the answer that our calculator finds. Reasonableness can tell us if a calculator's answer is one we can believe. The calculator's answer is exact. We use reasonableness to tell us if the calculator's answer is exactly right.

Questions from Teachers

1. In the books *Mathematics Their Way* and *Mathematics... a Way of Thinking*, children made up words like "yuck" and "zurkle" to describe the groupings for the different bases. Why are made-up words not used in this book?

The rule we teach our students for bases is that we cannot have any group representing the base number lying around loose. To apply the rule, all we have to know is the number for the base. Do yucks and zurkles have an easier rule than this? There is no need to add a different word or rule for every base when one will do for all. There is no need, but use yucks and zurkles if you want—they are fun and do no harm.

2. Doesn't it confuse students to see 10 and not have it mean ten? Wouldn't it be easier to have our students say " ten" whenever they see 10 written, or "eleven" whenever they see 11, regardless of the base? Saying "one cup, one" seems so artificial.

One of the differences between Asian and American children in the facility with which they learn mathematics is in the language they use to describe the numbers. In Chinese and Japanese, the words match the numbers. The number eleven in Chinese is said, "ten, one." Twenty is said, "two ten(s)." Seventeen is said, "one ten, seven." Once a Chinese or Japanese child knows the words for the numbers zero through ten, all the other numbers from 1 through 99 can be read.

Studies on understanding numbers have shown clear connections between how the numbers are said and how the numbers are understood. For example, when asked to represent the number 42 with counters, first-grade Japanese children in the study had no difficulty showing four groups of ten plus two more. Because the number 42 in Japanese is said "four ten(s), two," any child who can count to ten can arrive at the right number.

When the first-grade American children in the study were asked to represent the number 42, most depicted it as four counters next to two counters, or six counters altogether. Nothing in how American children learn to say a number gives a clue to what the number means.

Neither group of first graders had been exposed to place-value concepts at school. Yet the Japanese children could already demonstrate a basic concept of place value simply because of the names they used to describe their numbers.

Because no inherent understanding is present in the names Americans use for numbers, we teach our students to say "one cup, one" or "one ten, one" if the number is base ten. We teach our students in different bases so they will have an understanding of what the numbers represent. The best way to say the numbers is the way that translates understanding into words.

3. Is it really fair to let our students use calculators in school any time they want?

Our students' world is not the world in which we were raised. Our students do not need to spend the hours on calculations that were required when we were in school. If our students do not remember how to get an answer to a problem when they are outside of school, their calculators will do the remembering for them. Can we accept the changes in the world, or must we teach just as we were taught?

A fourth-grade student was told to bring both a dictionary and a thesaurus to school. The teacher had two separate books in mind. Instead, the student brought an electronic Language Master which contained a 200,000 word dictionary, a full thesaurus, a spelling checker, and a grammar guide.

The student expressed concern to his parent before taking the Language Master to school, "Will my teacher let me use this?" The parent's optimistic answer was, "Why not? I use my Language Master to help me whenever I have anything to write."

What would we do if this student came to our room? Was the parent's answer right? Would we let a child push buttons on a keyboard to find definitions for words? Or was the child's concern well founded? Would we feel our dictionary lesson had no value unless we heard the pages turn? Do we keep our teaching tied to the way we learned, or do we teach for the world in which our children live?

4. What do we do when there are report cards we must fill out?

Here are answers to the question from two different teachers with two different points of view.

First teacher:

Our district policy requires that we give every student a grade so I do as I am told. I fill out the report cards while giving them as little meaning as I can. I give grades of A in every subject that I can to every child in my class. Our district policy also says that no child can receive an A or B unless that child has passed a special test at grade level or above in reading and in math. Here, too, I do as I am told. I give the child the grades the district requires in reading and in math and give the child A grades in every other subject on the card.

When I meet with parents one-to-one at conference time to discuss the progress of each child, I explain my grading rationale. I focus each parent's attention on the portfolio I have collected of their child's work. I explain that I feel a portfolio is a truer representation of what the child has accomplished than any letters on a card.

Occasionally, I encounter a parent who does not like the fact that every other child has As on his or her report card. How can comparisons be made when everybody has all As? I remind this parent that learning is not a competition in my class. I help this parent focus instead on the quality of the learning that his or her child has done.

Second teacher:

Grades are required as a part of our report cards. Because it is a requirement, I have my students help me develop a set of rubrics to use. I want every child in my room to know exactly what is expected of them for each grade. I then have my students keep logs of their work and participate in deciding which samples of work go in their portfolios for grading later on. I also use observations, short traditional tests and student self-evaluations to collect more information to help with my evaluations. I send letters home to parents describing my grading process. I also spend part of my time at back-to-school night describing to parents how I assess my students.

I push all of my students to the level to which I think they are capable. I do not expect all children to reach the same level in the year they spend with me. We know that every child is different and learns at a different pace. My grades are based on my evaluation of each child's effort to achieve his or her maximum potential in my class. A top student doing only the minimum work to get by does not earn the highest marks. Struggling students who put their all into everything they do are rewarded for their persistence, even if they do not do as many problems as a faster but less motivated student does.

There is work that is acceptable and work that is not. I make this distinction very clear. Not all students get As. Quality (not the same quality for each child) and attitude play a role. I hold high expectations for all my students. I expect a lot from them no matter what their capabilities. I give my grades accordingly.

The rest of us:

If we had all the resources that we like, the teaching in our class might be much different than it is. The reality is that we teach the best we can with the materials we have at hand. An ideal world would require no grades in our class. If the class in which we teach is not ideal, we create any grading solution that we can invent to comply with the district rules while still keeping alive the belief that every child can learn.