Chapter 13

Advanced Multiplication and Division

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Before We Begin

How many times?...

Can we find the answer to this problem? Could our students?

$$\begin{array}{r} \mathbf{24} \\ \underline{\mathbf{x13}} \end{array}$$

Did I forget to mention that this is a problem in base five?

If the problem were in base ten, we could use our calculators to find the answer, or we could use the algorithms we memorized when we were students. Could we find the answer in base five?

Answers for beginning multiplication come from counting or from calculators. What did our lessons in beginning multiplication teach us about place value or about multiplying numbers in any base but ten? Calculators and algorithmic rules cannot guarantee our understanding of the arithmetic to use.

The *times* in 24 x 13 is not a uniquely mathematical term. As we were growing up, how often did we hear the questions:

How many times have I told you not to...?

How many times do I have to ask...?

And, as adults, how often have we repeated the "How many times...?" questions ourselves?

How many times? 24 x 13 means we have 24, 13 times. 24 x 13 in base five looks like this:

(illustration 13-0-1)
(Show eight rows, each with two cups and four squares in it. Each cup contains five squares. Group the eight rows so that there is a five row section and a three row section. Note in the caption that the five row cluster is one, zero rows. The point is to show that the eight rows is how 1-3 base five rows are represented.)

We would have less trouble picturing 24 x 13 if the numbers were in base ten.

(illustration 13-0-2)
(Show thirteen rows, each with two cups and four squares in it. Each cup contains ten squares. Group the thirteen rows so that there is a ten row section and a three row section. Note in the caption that the ten row cluster is one, zero rows.)

We found the answers for beginning multiplication and division problems by counting. We find the answers for advanced multiplication and division problems by using our knowledge of place value. When we know what the numbers represent, we can find an answer to any problem we might face regardless of the base.

Lesson One

Purpose	Learn what it means to multiply in any base. Learn to search for multiplication patterns within and between bases.
Summary	Students learn to make multiplication matrices for bases three through ten as they search for patterns in each new matrix.
Materials	Squares, blacklines for 11 X 11 blank matrices.
Topic	Base-three matrix made by the full class. Pairs of students work in pairs to make matrices fir bases four and five. For base six, students look at the previous bases for patterns that they can anticipate will appear. The process is repeated for bases seven, eight and nine. How much of base ten can be filled in from all the patterns seen in three through nine?
Topic	Base-four matrix started by the full class, finished working in pairs.
Topic	Base-five matrix in pairs, information shared collectively.
Topic	Base-six matrix made by looking at bases three, four and five.
Topic	Bases seven, eight and nine and, for some, sixteen.
Topic	Patterns is bases three through nine are used to fill in a matrix for base ten.

Matrix in a different base...

Teacher: You are all really good at making up addition and subtraction problems in different bases. Today we will see if we can create multiplication matrices in different bases, too.

We already know how to record square rectangles on a multiplication matrix. (*Beginning Multiplication and Division*, Lesson One (page 217)). **Let's see if we can record a square rectangle for base three.**

(illustration 13-1-1)
(Blank four by four matrix on the overhead with base 3 written at the top.)

Teacher: We will make the rectangles and record the answers directly on the paper just as we did when we made the matrices before.

(illustration 13-1-2)
(Square rectangle two across and one down on the overhead matrix. The rectangle is in the top row, with one edge touching the left edge of the matrix.)

Teacher: Please put this rectangle on your paper. Please check your neighbors and have your neighbors check you to see that you all have the rectangle in the top row and starting at the left side. Remember where we write the number for how many across?

Student: Above the last square in the rectangle.
Teacher: How many squares are there across?
Students: Two.
Teacher: Remember where we write the number for how many down?
Student: Next to that square.
Teacher: How many down?
Students: One.
Teacher: How many squares in the rectangle altogether? What is one times two?
Students: Two.
Teacher: We write the two beneath the farthest square across and down.

(illustration 13-1-3)
(The number two written outside the matrix above the second square over. The number one written outside the matrix beside the first square. Two written on the matrix beneath the square the farthest across and down. The square is shown slightly off center, so that it can be seen as moved but not yet gone.)

Teacher: Please put a rectangle three across and one down on your paper.
How many squares across is this rectangle?
Students: Three.
Teacher: What base did I say this is?
Students: Three.
Teacher: The rule for multiplying in base three is the same for adding and subtracting in base three. We can't have any of the base number. So, how do we write three in base three?
Students: One, zero.
Teacher: Show me on your trading board.

(illustration 13-1-4)
(Trading board with one cup on it with three squares in the cup.)

We have our students use their trading boards to group the numbers in their rectangles until they know the exchanges to be made without their boards. Trading boards help our students make connections between the place value they learned for addition and the place value they are using now.

Teacher: Where do we write the number for how many across?
Student: At the top of the paper, above the last square.
Teacher: How many down?
Students: One.
Teacher: Where do we write the number for how many down?
Student: Beside the first square. There is already a one written there.
Teacher: Okay. How many squares in the rectangle altogether? Remember that these numbers are base-three numbers. What is one times one, zero, base three?
Student: One, zero.
Teacher: Show me on your trading board.
Student: We already did.
Teacher: Okay. We write the one, zero beneath the farthest square.

(illustration 13-1-5)
(The number one, zero (10) written outside the matrix above the third square over. 10 written on the matrix beneath the farthest square across. The square moved is shown slightly off center, so that it can be seen as moved but not yet gone.)

Teacher: Please put a rectangle four across and one down on your paper.
How many squares across is this rectangle?
Students: Four.
Teacher: How do I write four squares in base three?
Students: One, one.
Teacher: Show me on your trading board.
Where do I write the one, one?
Students: Across at the top.
Teacher: How many down?
Students: You already have it written on the matrix. One.
Teacher: How many squares in the rectangle altogether? What is one times four in base three?
Students: One, one.

(illustration 13-1-6)
(The number one, one (11) written outside the matrix above the fourth square over. 11 written on the matrix beneath the farthest square across. The square moved is shown slightly off center, so that it can be seen as moved but not yet gone.)

Teacher: What size rectangle will I have you make next?

(illustration 13-1-7)
(Show an entire base three matrix.)

Teacher: Now, let's try a base-four matrix.
Please put a rectangle two across and one down on your paper.
Write the number for how many across.
Write the number for how many down.
What is one times two?
Students: Two.

(illustration 13-1-8)
(The number two written outside the matrix above the second square over. The number one written outside the matrix beside the first square. Two written on the matrix beneath the square the farthest across and down. The square is shown slightly off center, so that it can be seen as moved but not yet gone.)

Teacher: Please put a rectangle three across and one down on your paper.
How many squares across is this rectangle?
Students: Three.
Teacher: How do we write three in base four?
Student: You just write a three.
Teacher: How many down?
Students: One. It's already written there.
Teacher: How many squares in the rectangle altogether? What is one times three?
Students: Three.
Teacher: Write the number on your matrix.
Please put a rectangle four across and one down on your paper.
How many squares across is this rectangle?
Students: Four.
Teacher: How do I write four squares in base four?
Students: One, zero.
Teacher: Show me on your trading board.
How many down is already written there. What is one times four in base four?
Students: One, zero.
Teacher: What size rectangle will I have you make next?
Students: Five across.
Teacher: How do you write five across in base four?
Students: One, one.
Teacher: Write it on your matrix.
What is one times one, one in base four?
Students: One, one.
Teacher: Now, see if you and your partner can fill in the rest of the base-four matrix working together. As you work, I want you to look for patterns that might help you to see what numbers might come next in any of the rows or columns. If you finish base four, you may see if you can make a matrix for base five.

When most of the students have finished the base-four matrix, the teacher has all the students help fill in the base-four matrix on the overhead.

Teacher: Tell me what numbers you found for the ones column going down.
For the twos.
For the threes.
For the one, zero column.
For the one, one.

(illustration 13-1-9)
(Completed base four matrix.)

Teacher: What patterns can you see for the numbers on this matrix?

Some students fill in their base-four matrix quickly. Some students will not yet be finished by the time the whole class begins sharing. The matrix on the overhead ensures that all students can complete their matrix regardless of the speed at which they work.

(illustration 13-1-10)
(The base three and base four matrices, side by side on the overhead.)

Teacher: What patterns can you see that are the same on both of these matrices?

We ask our students to look for patterns, but we cannot always predict the patterns they will see.

Teacher: Now, if you have already started the base-five matrix, keep on working. If you have not started yet, you and your partner may begin. As you work, see how many patterns you can find that will help you know what numbers you can predict.

When most of the students have finished the base-five matrix, the teacher has all the students help fill in the base-five matrix on the overhead.

Teacher: Tell me what numbers you found for the ones column going down.
For the twos.
For the threes.
For the fours.
For the one, zero column.
For the one, one.

(illustration 13-1-11)
(Completed base five matrix.)

Teacher: What patterns can you see for the numbers on this matrix?

•□•□•

Teacher: Now, let's try base six. Let's see how much of a blank base-six matrix we can fill in just by looking at the patterns on your base-three, -four and -five matrices.
Before you raise your hand to tell me a pattern that you see, you must discuss what you see with your partner. Once you both see a pattern, both raise your hands. You both have to know what pattern you see, because you will not know which of you I will call on to describe it.

Some patterns are easy to see. Some are more difficult. Some we may see with this year's group of students. Some we may not see until next year's class. The particular patterns students see are not as important as learning to look. We can help our students learn to look by the questions that we ask:

What numbers go on the outside of the matrix, across the top and down the side?
What patterns can you see in any of the columns?
Are the patterns in the rows like any patterns in the columns?
What is the number in the lower right hand corner of each matrix?
Do you think the same number will be there for the base-six matrix?
Look at the numbers in the column that is one less than the base number. Is there a pattern for the numbers on the right side of the column? Is there a pattern for the numbers on the left side of the column? What might this pattern look like for the fives column in base six?
Are the patterns in the even-numbered bases the same as or different from patterns in the odd-numbered bases?

Teacher: Now, use your squares to check to see if your predictions for the base-six matrix were correct. For each column, make a few rectangles to see what numbers you really get. Also, use your squares to figure out the base-six numbers that we could not predict.

(illustration 13-1-12)
(Base six matrix completely filled in.)

As our students make the multiplication matrices for each succeeding base, they learn to connect the patterns in one base with the patterns in the next. As we work with our students we, too, see patterns and connections we might not have seen before. Did we know that 11 x 11 = 121 in every base but two? Were we sure that 10 x 10 = 100 for bases other than ten? If we had seen the pattern in base ten for the multiples of nine before, did we know that nine's pattern is shared in common with all multiples that are one less than the base?

Our students work together to fill in as much of the base six matrix as they can from the patterns that they see. They check their predictions with their squares. They then continue the process of filling in the matrices for bases seven, eight and nine. Students who work faster than the rest may try their hand at filling in a matrix for base sixteen.

Why base sixteen? Computers use it all the time. Computer programmers find uses for it, too. How do we make a matrix for base sixteen? The rule for bases is that we cannot have any of the base number lying about. Anytime we have sixteen squares, we put the squares into a cup. How do we write the numbers for sixteen when our numbers stop at nine? We use letters when our numbers run out.

(illustration 13-1-13)
(Match squares and cups with the appropriate base sixteen number from one to about thirty-two (20, base sixteen). 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 1A, 1B, 1C, 1D, 1E, 1F, 20.)

(illustration 13-1-14)
(A base sixteen matrix seventeen squares by seventeen squares, completely filled in.)

When most of our students have finished bases seven, eight and nine, we work together as a class looking for the patterns in base ten.

Teacher: Now, let's see how much of a blank base-ten matrix you can fill in by looking at the patterns for the numbers on all your other matrices.
Before you raise your hand to tell me a pattern that you see, you must discuss what you see with your partner. Then both of you raise your hands together.

Our students have created the base-ten matrix many times before, but they did not call their matrices "base ten." Now we ask our students to connect the patterns they have seen in base ten to every other base. Our students may have thought they had seen all the patterns they could see in the matrices they created for beginning multiplication, but, in mathematics, as in life, there is always more for us to learn.

Lesson Two

Purpose	Learn how to represent concretely a multiplication problem larger than a matrix shows. Learn how to multiply in any base.
Summary	We give our students larger multiplication problems to solve. We help our students apply what they already know to the new problems that they face. Once they understand the process, they use dice to create problems of their own to solve.
Materials	Chalkboards, squares, cups, bowls, trading boards, dice, multibase blocks (if available).
Topic	Multiplication problems in base five.
Topic	If available, multibase blocks and multiplication problems in different bases.

Squares and cups and bowls...

How many seconds are in a day? Sixty seconds in a minute times sixty minutes in an hour times twenty-four hours in a day gives eighty-six thousand four hundred seconds in a day.

The numbers on a multiplication matrix are the answers to small problems. Our students cannot solve every problem they encounter by laying out as many squares as the numbers represent. Even if we had enough squares to match the seconds in a day, how much space would the squares consume?

$$\begin{array}{r} \mathbf{24} \\ \underline{\mathbf{x13}} \end{array}$$

Teacher: I would like you and your partner to solve this base-five multiplication problem and write your answer on your individual chalkboards.
Student: But we don't know how.

Teacher: Can you solve this one?

11
x11

Teacher: I can also write it this way:

11_____
11 |

Students: Yes. It's one, two, one.
Teacher: I didn't even tell you what base this is. How do you know its one, two, one?
Students; It's always one, two, one.
Student: Except in base two. You can't have any twos in base two.
Teacher: How would you solve this one?

12
x11

Teacher: I'll write it the other way, too.

12_____
11 |

Student: What base?
Teacher: What base would you like it to be?
Student: Base three.
Teacher: Okay, it's a base-three problem. How would you find the answer?
Student: Make a rectangle that's one, two across and one, one down.
Teacher: How many squares do I need to make one, two across in base three?
Student: Five.
Teacher: And how many down is one, one?
Student: Four.
Teacher: Then, to find the answer to this problem, what size rectangle would you make?
Student: Five across and four down.
Teacher: Make it please, and tell me what the answer is for base three.

(illustration 13-2-1)
(Show the square rectangle five across and four down. Then show the squares on a trading board grouped into two bowls, each with three cups in them, and two squares.)

Students: Two, zero, two.

12
x11
202

12_____
11 | 202

Teacher: One way to show the problem is to make a rectangle. But we can also show it another way. How many times does this problem say we have one, two?
Students: One, one times.
Teacher: First, let's put out one, two.

(illustration 13-2-2)
(One cup and two squares in base three.)

Teacher: How many times do I have one, two here?
Students: One time.
Teacher: How many times do I have to put it out to have it one, one times?
Students: One, one times.
Teacher: But how many times is that? How many sets of one, two do I have to make? One, one is a number grouped for base three. What number groups to one, one in base three?

Student: Four.
Teacher: Then I have to put one, two on my trading board four times.

(illustration 13-2-3)
(One cup and two squares set out four times on a trading board. The four separate sets are placed on the trading board as they would be for a four row addition problem.)

Teacher: Now I have one, two on my trading board one, one times. To find out the answer, I add all the one, two rows together.

(illustration 13-2-4)
(Add the problem in the previous illustration. First half of illustration, all the squares and cups at the bottom of the trading board. Second half, the appropriate exchanges made. The answer on the board is two bowls and two squares.)

Teacher: Either way we did it, we got two, zero, two as the answer. Now, I would like you and your partner to solve this base-five multiplication problem and write your answer on your individual chalkboards.

<u>24_____</u>
13 |

24
<u>x13</u>

I have written the multiplication problem two different ways.
Use the first way if you use a rectangle to find the answer, then record the problem and the answer that you find in the sideways L. The sideways L shows the numbers for the rows and columns. The answer tells how big the rectangle is.
Record your answer the second way if you add the two, four; one, three times.

The answers on the chalkboards tell the teacher if the class collectively understands the process of finding answers to multiplication problems using squares, cups and bowls. If the class understands, then the students can begin making up problems on their own. If too many students need assistance, the teacher works the problem on the overhead aided by the students who understand.

Students may also create problems using dice just as they did in *Advanced Addition and Subtraction* (page 302). The teacher may decide to have the students create all the problems in base five, or have each new problem created for a different base, or let the students themselves select the base or bases they will use. The teacher may also decide to have the students alternate between solving problems with rectangles and solving problems on their trading boards.

When we understand place value and what the numbers represent, we can find an answer to any multiplication problem that we face.

Multibase blocks...
Multibase blocks give our student another opportunity to look for patterns and make connections. The blocks enable our students to see the numbers in their problems formed a different way.

(illustration 13-2-5)
(Show the complete process of creating a multiplication problem for multibase blocks. Accompany the illustrations with descriptive captions. First, show the blocks, dice trading board and recording paper. Roll for the top row. Roll for how many times the row is to be put out. Show the numbers written on the recording sheet. Show the row set out that many times on the trading board or on the table top if the trading board is not large enough. Show all the exchanges being made. Show the answer written on the recording sheet.)

Lesson Three

Purpose	Learn techniques for finding answers to the multiplication problems likely to be on the end-of-year standardized test.
Summary	If our students will not be permitted to use calculators or materials on their year-end test, we teach them how to calculate answers for the test.
Materials	Paper.

Topic	An algorithm for multiplication is taught.

Practical ways…

Squares, cups and multibase blocks are not practical ways to find answers to very big problems—they are practical ways to find what the answers to very big problems mean.

Here is a problem in base ten:

$$\begin{array}{r} \mathbf{25} \\ \underline{\mathbf{x13}} \end{array}$$

The most reasonable way to solve this multiplication problem is with a calculator. Could we really let our students solve all their problems in a reasonable way? How can our students pass the end-of-year test if we do not teach them the methods the test might require?

Here is how most of us learned to solve problems:

$$\begin{array}{r} \mathbf{25} \\ \underline{\mathbf{x13}} \\ \mathbf{75} \\ \underline{\mathbf{250}} \\ \mathbf{325} \end{array}$$

The way we were taught is not the only way to find answers when the use of a calculator is not allowed.

$$\begin{array}{r} \mathbf{25} \\ \underline{\mathbf{x13}} \end{array}$$

Is the same as:

$$\begin{array}{r|l} & \underline{\mathbf{25}} \\ \mathbf{13} & \end{array}$$

We can use our knowledge of place value to write the numbers outside the L and change the L into a matrix-like box:

	20	**5**
10		
3		

We fill in the boxes using the same down-and-across patterns that we used for all the other matrices:

	20	5
10	**200**	**50**
3	**60**	**15**

We add the numbers in the boxes down:

	20	5
10	200	50
3	60	15
	260	**65**

Then we add the totals for the numbers down, together:

	20	5
10	200	50
3	60	15

260 + 65

$$\begin{array}{r} \mathbf{260} \\ \underline{\mathbf{+65}} \\ \mathbf{325} \end{array}$$

Or, we add the numbers in the boxes across:

	20	5	
10	200	50	**250**
3	60	15	**75**

Then we add the totals for the numbers across, together:

	20	5	
10	200	50	**250**
3	60	15	**+75**
			325

Or, we can add the numbers in the boxes in one long column:

	20	5	
10	200	50	**200**
3	60	15	**50**
			60
			+15
			325

Or, we can use the lattice multiplication described in *Mathematics... a Way of Thinking*, Lessons 9-7 through 9-18.

The purpose of the way we choose is to prepare our students for the abstract methods that may be required on the year-end tests. Whether we teach our students the way we were taught, use the matrix-like boxes, or use the lattice, our students use their calculators to check the accuracy of the answers that they find.

Lesson Four

Purpose	Learn to look for patterns in multiplying numbers with zeros at the end. Learn how to find reasonable answers for multiplication problems that are large.
Summary	Students use their calculators to fill in worksheets that they then examine for patterns for multiplying numbers with zeros on the right. Students use the patterns to learn to estimate reasonable answers for large problems.
Materials	Blackline worksheet for 1, 10, 100, 1000; chalkboards; calculators.
Topic	Multiplying by 2, 20, 200, 2000 and so on. Patterns sought.
Topic	Reasonableness for answers to large problems.

True stories...

League organizer: Okay, we have eleven teams signed up and five people are on a team. How many jerseys are we going to need?
Coworker. I don't know. I was never any good at math.
Organizer: Well, I always try to make the problem simpler for myself. Let's say there are only ten teams. How many people is that?
Coworker: I don't know.
Organizer: What's five times ten?
Coworker: (Hesitates.) Twenty-five?

Doesn't everyone know that five times ten is fifty? We may not have been asked to look for patterns when we were in school, but there are some patterns we might have discovered on our own.

1 x 10 = 10
2 x 10 = 20
3 x 10 = 30
4 x 10 = 40
5 x 10 = 50

Manufacturer to newly hired clerk: We will need to know the individual cost of each item in this kit and the total wholesale cost of one-thousand for each item. Get the costs, do the multiplication and give the numbers to me when you're done.
Newly hired clerk: (Begins writing out separate problems for each item multiplied by 1000.)
Manufacturer: (Checking the clerk's work in progress.) Good Lord! Don't you even know how to multiply by a thousand? Don't you even know to just add zeros?
Soon-to-be-fired-clerk: You said to multiply the prices. This is how to multiply. What zeros am I supposed to add?

What is a thousand times any number? That number with three zeros added to the end. Not every problem that we see is a different problem to be solved.

1 x 1,000 =	**1,000**
15 x 1,000 =	**15,000**
27 x 1,000 =	**27,000**
42 x 1,000 =	**42,000**

What is a thousand times any decimal number? That number with three zeros added to the end. The decimal has a pattern, too. For multiplying prices, the decimal stays two numbers from the right.

1.19 x 1,000 =	**1,190.00**
1.57 x 1,000 =	**1,570.00**
2.75 x 1,000 =	**2,750.00**
14.21 x 1,000 =	**14,210.00**

Are the patterns for multiplying by 20, or 300, or 4,000, patterns that we know? If we have seen the patterns, then we make sure our students see them, too. If we have not seen the patterns, then we will see them as our students do.

Counting all the zeros...

(illustration 13-4-1)
(Multiplication Patterns worksheet from blackline masters.)

Teacher: We will use this worksheet to see if we can discover some patterns for multiplying numbers. You may use your calculators to find the answers. If you choose to work with a partner, please write the names of everyone in your group on your paper.
I will give you an example of how to use this worksheet before you begin working on your own. What does this sentence say?
Students: Choose a number for the blanks.
Teacher: The sentence means choose one of these numbers, one through nine, and then put that one number in all of the blanks in the top row. I'll choose the number two.

(illustration 13-4-2)
(The 2 on the worksheet circled and 2 written in each of five blanks along the top row.)

Teacher: (Pointing to each number in turn in the top row) As I point to each number in this top row, please read it to me. Remember how to read the commas.
Students: 2, 20, 200, 2,000, 20,000..
Teacher: What does this column heading say?
Students: Made-up numbers.
Teacher: In this first column, we pick a number we will multiply. I'll write in twenty-five.

(illustration 13-4-3)
(The illustration from above with 25 written in the top left-hand cell.)

When the students make up numbers to multiply for their own worksheets, some will pick small numbers, some will pick large. Large or small, the pattern is the same.

Teacher: What number do you think you are to multiply twenty-five by to get the first answer?
Student: Two.
Teacher: Multiply twenty-five times two on your calculator, write your answer on your chalkboard and put your chalkboard face down on your desk.
Okay, everybody hold up your boards. The answer that I see the most often is fifty.
What number do you think you multiply twenty-five times next?
Students: Twenty.

The students multiply 25 by 2, then 20, then 200, then 2,000, then 20,000 as the teacher records the answers on the overhead.

(illustration 13-4-4)
(The illustration from above with 50 written in the cell immediately to the right of the 25, and the first rows of answers filled in completely.)

Then a new made-up number is selected and the process is repeated. Once the students understand how to use the worksheet, they select their own number and, working alone or with a partner, use their calculators to find all the answers that they can. Students who work more quickly take another worksheet, choose a different number for the blanks, and continue the process of making up numbers to multiply.

As the students work, the teacher asks:

Are there any patterns that might tell you what numbers to expect for each new column?
Is the pattern that you see the same for any number that you multiply?
Are there any numbers for which your pattern does not work?
Does counting all the zeros give you a hint of the numbers to expect?

Monica and Andrew revisited...

In *Beginning Addition and Subtraction* (page 147), the story was told of third-grade students asked to find an answer to a problem on their calculators:

$$\begin{array}{r} \mathbf{932} \\ \underline{\mathbf{x361}} \end{array}$$

Monica's answer was 27, Andrew's answer was 531,441. Neither Monica nor Andrew knew how to find answers to large problems on their calculators. Neither knew how to use a pattern for counting zeros to predict what a reasonable answer might be.

The problem can be rounded down to:

$$\begin{array}{r} \mathbf{900} \\ \underline{\mathbf{x300}} \end{array}$$

Or up to:

$$\begin{array}{r} \mathbf{1{,}000} \\ \underline{\mathbf{x400}} \end{array}$$

For the problem rounded down, what is 9 times 3? How many zeros should there be in the answer to this problem? The Multiplication Patterns worksheet tells us to expect four. Twenty-seven with four zeros is 270,000.

$$\begin{array}{r} \mathbf{900} \\ \underline{\mathbf{x300}} \\ \mathbf{270{,}000} \end{array}$$

For the problem rounded up, what is 1 times 4? That's an answer not too hard to find. How many zeros in the answer to this problem? Five is the number of zeros we expect. Four with five zeros added to the end is 400,000.

$$\begin{array}{r} \mathbf{1{,}000} \\ \underline{\mathbf{x400}} \\ \mathbf{400{,}000} \end{array}$$

We expect 932 x 361 to have an answer between 270,000 and 400,000. What is the answer if we multiply?

$$\begin{array}{r} \mathbf{932} \\ \underline{\mathbf{x361}} \\ \mathbf{336{,}452} \end{array}$$

Is Monica's 27 or Andrew's 531,441 an answer in-between? We can teach both Andrew and Monica better ways to seek out reasonableness. Knowing when an answer is reasonable comes from:

Understanding place value.
Making hard numbers into simple ones.
Counting out the zeros.

When our students can see patterns for multiplying by 1, 10, 100, 1000 and so on, we ask them to use the patterns to estimate answers to large problems before they multiply.

Lesson Five

Purpose	Learn to apply multiplication skills.
Summary	With our students, we look for real problems to solve that use multiplication.
Materials	Writing paper.
Topic	A problem a day is enough to pose. A problem is posed, small groups of students discuss and write down ways it might be solved, then students share their ways with the class.
Topic	Problem posed, discussed, solved, solutions shared.
Topic	Another problem is posed.
Homework	Our students search at home for multiplication problems that occur naturally. Parents join in the search.

How many?...

Once our students understand how to find answers to multiplication problems, with their calculators or without, we give them problems to solve that ask them to use their multiplying skills.

How many children are in our school? Shall we add all the numbers of all the students in each of the rooms, or can we make an estimate that is close enough? Can we guess an average number of students in each room and multiply by the number of rooms we have? What is an average number anyway?

How many books are in our school library? Shall we count every book to find out, or is there a way to come close enough to the number that would not take us as long? Can we count all the books on a shelf or two or maybe even three to guess the most common number on most shelves? Is this most-common number anything like an average number? How many books on a shelf? How many shelves in a book case? How many book cases in the library? How close might our number be to the actual number of books on the shelves? Would our number be close enough, or would we feel compelled to count every book just to find out? How many books are there for each child in the school? Maybe the librarian can tell us how close our number is to what she thinks is true.

How many sheets of writing paper in a full carton? How many sheets in each ream? How many pages for each child in our room? Can we ask a question that might mean dividing when we have not taught division yet? Must we ask only the questions that match the lesson that we teach, or do we want our students to think about what the numbers mean? We can ask our students to discuss with each other the methods they might use.

How many people will the theater hold? Shall we count every seat, or is there an easier way? How many seats in each row? How many rows in each section? How many sections in all? What do we do with all of the numbers for all of the chairs, sections and rows?

How much multiplication is left for a student to learn once the process of multiplication is understood? How much language is left for any child to learn once the child has learned to speak?

One is enough...

The textbooks and workbooks have pages and pages of problems for our students to solve. How many problems should we invent each day to give our students all the practice that they need? One is enough. One from the teacher. One from a student. One that a child brings from home to explore.

Teacher: How many of you get an allowance? Raise your hands.
Students: (Many hands raised)
Teacher: For those of you who get an allowance, how many get your allowance once a week?
Students: (Almost as many hands raised)

Teacher: What I would like you to do today is figure out how much allowance you get for an entire year. How many weeks are in a year?

If no one knows, the teacher volunteers the answer fifty-two.

Teacher: How many months?

If no one knows, the teacher volunteers twelve.

Teacher: Work with your partner to figure out how much money you receive in a year. If neither you nor your partner get an allowance, figure an answer for the allowance you think you should get. If you make up an allowance, then make up how often your allowance is paid.
When you find a total, write down on paper how you figured out your answer. In a while, we'll discuss together all the different ways everyone is class thought to use.

As our students work, we walk around the room observing all the different approaches. If no one has any idea of what to do, we lead the class in a discussion of the ways that might be tried. If only a few can think of ways and others need a hint, we share aloud the ways we see that are being tried.

Any way a team of students uses to find an answer is acceptable to us. The ways each team of students uses are written up for sharing; then each group shares aloud the method used and the answer found. All the other groups join in to discuss the reasonableness of the methods and the answers.

Thinking, discussing, calculating, writing, sharing. All of this takes time. How many problems do our student have to do each day to receive all the practice they need? One is quite enough.

Knowing when to use it...

Knowing how to do it is not the most important thing about multiplication. The most important thing is knowing when to use it. We want our students to know when to use the operations that we teach, so even when our students practice multiplication, the problems we give sometimes need the skill of multiplying and sometimes do not.

How many pencils in our class?
How many pencils in the school?

The answer to the first question may be found by adding. The answer to the second may mean we need to multiply. Neither question includes within it the method to be used. The problems that we give ask our students to think of how the problems might be solved.

How many pieces of paper get used in our class each day?
How many pieces of paper get used each day in the school?

Pencils in the class can be counted within minutes. How long before we know the pieces of paper used each day? Is every day the same? Not every problem needs to have an answer that is quick to find.

The questions we ask...

We know that multiplication is adding the same number over and over again. Anytime we see numbers that are the same, we know we have a multiplication question waiting to be asked. Where are there numbers that are the same?

Pencils stored in little boxes in the school's supply room.
Construction paper available to use.
Cafeteria trays. Servings of each food.
Numbers of Unifix Cubes or Power Blocks in our room.
Heartbeats in a day, week, month, or year. Heartbeats in a life.
Grocery stores with items grouped on the shelf.
Costs of all the items. What would it cost to buy all the items in a clothing store? A toy store?
Storage shelves on the sides of delivery trucks for water bottles or bread.
Parking lots with rows of cars or empty spaces.
Any place where people sit or stand in rows.
How many glasses of milk will we drink each year if we have a glass or two each day?
How many French fries are in a fast-food serving? How many servings do we eat each year?
What is the weight of all the food we eat in a day or month or year?
Measurements of area and volume are already multiplication problems. What other kinds of measurement give us questions we can ask?
Numbers from sports: How many games altogether are played by all the teams in the league?

How many squares on the wall? How many squares cover the floor?
If we all have two parents and four grandparents and eight great grandparents, how many great, great, great grandparents do we have? How far back does our family tree go?

We look for situations for our students to investigate. We teach our students to join us in the search.

A workbook page of problems carries the message that quantity alone will produce the learning we desire. What do we learn by doing pages of problems when no one problem on the page involves an understanding different than another? If we understand how to do the problem, do we understand one problem any better if we clone the problem thirty times? If we do not understand, does repeating our ignorance thirty times cause our ignorance to disappear?

When we give our students a problem each day or two to reason out, we teach our students that problem solving and thinking go hand in hand. Numbers are tools for solving problems. Mathematics is something to think about.

The point of the questions that we ask is to connect the process of multiplying with the uses of multiplication. If we cannot think of ways to use mathematics, then what is the point of all we teach? If we cannot show our students applications for their learning, they will learn it for the test and then forget. The point of all the questions is to help our students see the mathematics in their lives.

Don't forget to write...

As our students prepare to share solutions with their classmates, we ask them to write down what they have done. Writing can mean words on lines. Writing can mean drawings, too.

School divides learning into compartments, but learning is not compartmentalized. Writing is not something we reserve for an English class. We use the skill of writing to communicate our thoughts. We help our students make the connection between mathematics and the writing skills we teach.

Writing helps our students remember the methods that they used to find their answer while waiting for their time to share. Writing also gives us a written record to add to each student's portfolio of work.

Lesson Six

Purpose	Learn what it means to divide in any base.
Summary	We teach dividing techniques. Our students then use dice to create division problems for themselves.
Materials	Squares, cups, bowls, trading boards, dice, calculators.
Topic	Divide in base five, then base four and base six.
Topic	Divide in base ten. Calculators check answers.
Topic	If available, divide with multibase blocks in different bases.

Not just multiplication in reverse...

In *Beginning Multiplication and Division* (page 223), we asked three questions for the rectangles we made with squares:

1. We know the squares across and the squares down. How many total squares? (Length times width equals area.)

$$\begin{array}{r|l} & \underline{\mathbf{y}} \\ \mathbf{x} & \mathbf{?} \end{array}$$

2. We know the total squares and the squares across. How many squares down? (Area divided by width equals length.)

$$\begin{array}{r|l} & \underline{\mathbf{y}} \\ \mathbf{?} & \mathbf{z} \end{array}$$

3. We know the total squares and the squares down. How many squares across? (Area divided by length equals width.)

$$\begin{array}{r|l} & \underline{\mathbf{?}} \\ \mathbf{x} & \mathbf{z} \end{array}$$

We used these questions to make a connection between multiplication and division. Multiplication is the method we use to answer question one. Division is the method we use to answer questions two and three. The connection is a useful one to make, but once the connection has been made, how often do our students need to know the answer to questions two or three?

Multiplying to find area is common in our lives. Houses are advertised for sale by the square foot. Rugs are sold by the square yards needed to cover up the floor. Fertilizer is sold by the amount necessary to cover square areas of lawn. Forest fires are measured by the square miles that they burn.

We find area by multiplying length times width. Do we ever use an area that we know and the length of just one side to find the unknown length for any other side? If we do not know a length, dividing is not a common method that we use to find it out. Measuring is the method we employ. We would not use the methods found in questions two and three.

We multiply to calculate the cost of the tank of gas we buy—the number of gallons times the price of each gallon gives us total cost. If we developed math amnesia and could not remember how to multiply, we could find the total cost by adding the price of each gallon as many times as we had gallons in our tank. We divide to calculate the miles we get on all the gallons used. Total miles divided by the gallons used gives us miles traveled per gallon. If amnesia struck again, would we subtract to find the miles per gallon? We would not.

When we add repetitively, the answers that we find are no more complicated than the numbers that we add. Adding three to three to three to three or taking three times four gives us twelve. We do not get an answer with a fraction or a decimal or a remainder of any other kind. As any child knows, dividing is much more complicated than adding three four times. When three children have seven cookies to share among themselves, the decisions they must make about the seventh cookie is what makes division the untidy operation that it is. Division is not just multiplication in reverse.

Our students know about division. They have been dividing things among themselves since before they were old enough to come to school. Division is used by children everywhere for fairly sharing what they have. As much as our students know about division, they do not know as much as we. We use our lessons on division to teach our students more efficient methods for determining a fair share. We give them a clearer understanding of the reasons why the methods work. When we show our students how to use their calculators for division, we teach them the meaning of the funny answers that their calculators sometimes give.

Basing our curriculum on mediocrity...

When we were in school, we spent years learning and reviewing how to do long division. Now, calculators find our answers for us in a flash. Can we remember the last time we did a long division problem that we encountered in our daily life the way we were taught in school?

There is much of value we can pass on to our students, and very little time each day to pass it on. Why, then, do we waste our precious teaching time drilling our students on the long division algorithm when we know it is a skill they will not use? Is the only reason we teach the long division algorithm because the algorithm will appear on the year-end test?

A *Wall Street Journal* article reported on a Boston College study:

> The standardized math tests most U. S. students take are too simplistic and they are causing teaching problems and impeding student progress...
>
> A recent study, conducted by researchers from Boston College, found that the most commonly used standardized and textbook exams assess mostly low-level skills and fall far short of the current standards recommended by math curriculum experts. The tests emphasize and mutually reinforce low-level thinking and knowledge and were found to have an extensive and pervasive influence on math instruction nation wide...
>
> The study examined the math sections of six standardized tests that dominate the school-testing market nationwide: the California Achievement Test, Comprehensive Test of Basic Skills, Iowa Test of Basic Skills, Survey of Basic Skills of Science Research Associates, Stanford Achievement Test and Metropolitan Achievement Test.
>
> Because many school systems place strong emphasis on the tests, teachers often tailor their instruction and curriculum to preparing for the exams. Moreover, the study found that teachers with high percentages of minority students are most likely to gear their teaching to the tests. Thus, minority students may be most likely to receive instruction that is dominated by low-level factual instruction and places relatively little emphasis on problem solving skills...

> In math, the study said, 95% of the items on both the standardized and textbook tests measured recall of information, computation and the use of algorithms and formulas in routine problems. Only 3% assessed sophisticated conceptual knowledge and just 5% gauged high-level thinking skills.
>
> "Math Tests Are Too Simplistic," *Wall Street Journal*, October 16, 1992.

When we base our curriculum on the year-end tests, we are basing our curriculum on mediocrity. Is doing well at mediocrity so worth our students' time? When do we stop teaching for the test and start teaching for our students' needs?

The long division algorithm is a skill made obsolete by time. There is no need to teach it anymore. We teach our students to understand what it means to divide. Once our students understand, they use their calculators to get the division answers that they need.

Dividing large numbers...

Teacher: Today we will practice dividing large numbers. We will be working in base five. Please put two bowls, three cups and four squares on your trading board. Make sure you have the correct number of squares for base five in the cups and bowls.

(illustration 13-6-1)
(Three column trading board with two bowls, three cups and four squares on it.)

Teacher: Please tell me how many squares you have altogether.
Students: Four.
Teacher: How many squares do you have altogether in your two bowls? Check with your partner to see if you both agree on the number.
Students: Fifty.
Teacher: How many squares do you have in your three cups?
Students: Fifteen.
Teacher: How many squares in your square column?
Students: Four.
Teacher: What is fifty plus fifteen plus four? You may use your calculators to help you find out.
Students: Sixty-nine.
Teacher: Whenever I ask you to tell me how many squares you have altogether, I want to know the total number of squares on your trading board whether the squares are in the squares column or the cups column or the bowls column. How many squares do you have altogether?
Students: Sixty-nine.
Teacher: Okay. Now please divide all your squares into four groups, so that each group has the same number as each of the other groups. Work with your partner to find a way to divide up all the squares. It is okay if you decide to dump the squares out of the bowls and cups to divide them up, but if you do, once you have finished the dividing, please put the squares in each group back into the appropriate cups and bowls. Put any left over squares aside.

(illustration 13-6-2)
(Show at least two different ways students divide to find the answer. Use the ways described in *Mathematics... a Way of Thinking* beginning on page 134. Include within this illustration the answer to the number in each group, by forming the squares into four clusters of three cups and two squares each, with a single square left over.)

The teacher asks each team of students to share its solution with the class. There may be as many ways the students find to divide as there are teams of students in the room.

When we were in school, division grew increasingly difficult as the numbers grew larger. Division involved estimating quotient numbers to use, multiplying and subtracting at each step and writing down the numbers in the proper lines. There were so many operations to perform to get the answers to come out right, division was among the hardest things we had to do. For us, division was another set of rules to memorize. For our students, division is another use of numbers that they understand.

Our students found the answers to their beginning division problems by counting out the squares. Now our students learn to group the squares counted out into cups and bowls. Our students already know how to divide. They also already know how to group and ungroup squares. The refinement that they learn now is how to write the answers that they find.

Teacher: The division problem you just solved is two, three, four, base five divided by four. It is recorded like this.

4 | 234

Or like this.

234 ÷ 4

The answer to the division problem is how many bowls, cups and squares were in each group. How many bowls, cups and squares do you have in each of your groups?
Students: Three cups and two squares.
Teacher: Were there any squares that did not fit into the groups?
Students: Yes.
Teacher: How many?
Students: One.
Teacher: That square is part of the answer, too. There were three cups and two squares in each group with one square left over. We write the answer like this.

32 1/4
4 | 234

Or like this.

234 ÷ 4 = 32 1/4

We record the remainder the same way you will when you get to high school. The one means how many were left over. The four means how many groups there were altogether.
Let's try another base-five division problem. Put four bowls, two cups and no squares on your trading boards and divide all the squares into three groups.

The teacher continues to make up problems for the students to divide until the class as a whole demonstrates an understanding of dividing large numbers in base five.

Teacher: Now, I would like you to begin making up your own division problems with dice. I will give you an example of what I mean. The first roll of my die tells me how many squares to place in the squares column.
What number did I roll?
Students: One.
Teacher: Then how many squares do I put in the square column?
Students: One.
Teacher: What number did I get for my second roll?
Students: Three.
Teacher: What do you think I will have the three represent on my trading board?
Students: Cups.
Teacher: Next roll.
Students: One.
Teacher: Where will I put the one?
Students: One bowl.
Teacher: Now what?
Student: Roll to see how many groups.
Teacher: Okay. What did I get?
Students: Two.

(illustration 13-6-3)
(A trading board with one square in the squares column, three cups in the cup column, and one bowl in the bowl column. A die with two showing placed next to the trading board.)

Teacher: For this last roll, I leave the die on my table showing two, so I can remember by which number I am dividing. Once I have rolled to create my division problem, I do all the dividing and see what answer I get.
As you and your partner work, please remember to record on paper all the numbers for the problems you create so I can see how you are doing. You may use either of the two ways to record division problems that I showed you.

If you roll a zero for the number of groups, please roll your die again. Let's not worry about having to divide your bowls, cups and squares into zero groups today.

As our students create their own division problems, we walk around the room observing the level of understanding present. When students understand the process of dividing in base five, we have them demonstrate their understanding to us again in base four or base six. We may decide to use base three as well. We introduce base ten only after our students understand the process of dividing in any other base we choose.

When our students begin dividing in base ten, they use their calculators to check each problem that they solve. Mathematics is connections. We connect the materials to the machine.

As was true before...

(illustration 13-6-4)
(A division problem set up with multibase blocks. Show all of the steps in solving the problem.)

As was true for our lessons on addition, subtraction and multiplication, we use squares, cups and bowls to help our students make the connections between the numbers in the problems and what the numbers represent. As was also true before, if we have multibase blocks to use, we feel pleased with our good fortune. Multibase blocks continue to be the perfect material to use between lessons with squares and lessons that use the calculator.

Know that they know...

The point of this lesson is to tie all the learning that has gone before to an understanding of division. Our students already understand the process of dividing. As the numbers to divide grow larger, our purpose is to ensure that our students continue to feel as comfortable with much bigger numbers as they already are with smaller ones.

For many of us, division was the worst of the arithmetic operations we faced. Our goal for our students is that they know mathematics well and that they know that they know. With understanding in their minds and calculators in their hands, this generation of students will find it inconceivable that there were ever people who could be terrible at math.

Lesson Seven

Purpose	Learn to use the arithmetic skills we have.
Summary	We look for real problems that use division and any other arithmetic skill. Our students look, as well, and keep a written record of the problem solving steps they use.
Materials	The materials relate to the problems found.
Topic	A single sandwich.
Topic	The daily life of a child.
Topic	Analysis.
Topic	Averages of all kinds.
Homework	Our students search at home for division problems that occur naturally. Parents join in the search.

Opportunities without limit...

We know the pattern for the operations by now. We teach the concept with materials, then introduce the calculator as a checking device. Once the connection between the materials and the calculating machine has been made, we give our students problems from the world around them. We connect the concept to the realities of our students' lives. We make mathematics real. Opportunities exist without limit when we let our imaginations run free.

A single sandwich...

Teacher: What does it cost your parents to make a single peanut butter and jelly sandwich for your lunch?

Where might this one question lead?

How much is a loaf of bread?
Do all loaves cost the same?
What is the price of a single piece of bread?
What do we divide into what to find the cost of one piece?
What is the cost for two pieces of bread?

How much does a jar of peanut butter cost?
Does all peanut butter cost the same?
How much of a jar's contents are spread on a single slice of bread?
Which numbers do we divide to find the cost of the peanut butter on a single piece?
How much does a jar of jelly cost?
Do all jars of jelly cost the same?
How much of the jar's jelly contents are spread on a single slice of bread?
Which numbers do we divide to find the cost of the jelly that we spread?
Would bigger jars of peanut butter or jelly have cost the same per slice?
Does it make any difference what store the jars come from?
If we put our sandwich in a plastic bag, what is the cost of the bag we use?
What other kinds of sandwiches are in the lunches in our class?
Can we calculate the cost of all the different items used to make these sandwiches?
What is the most economical sandwich to make?
What did our entire lunch cost?
Is a lunch from home more or less expensive than the lunch we buy in the cafeteria?
Can we answer the questions we ask in a day? Or, do we need more time?
What kinds of arithmetic are we using to find our answers?
What other kinds of mathematics are involved?

We do not ask our students every question that might be asked. We let the level of our students' interests be our guide. We can stop at finding out the cost of just one sandwich, or we can keep on asking until we find ourselves contemplating sandwich making as our career. There is much more math in a lunch bag than on any workbook page.

The daily life of a child...

What are the costs in the daily life of a child? What arithmetic skills do we use to calculate these costs?

At home, each child leaves a trail of costs throughout the day:

Toothpaste and toothbrush.
Water down the drain.
Soap for washing hands and face.
Toilet paper used.
The morning meal's food and drink.
Paper towels for wiping up the spill.
The lunch packed up for school.
Gas for the car ride to and from school.
Snacks after school.
The evening meal complete with a dessert.
The piece of pencil used to write the spelling words ten times.
Bubbles in a bath.
Soap for doing dishes and for laundry.

At school is a trail of costs of all the things a child consumes:

Pencils.
Crayons.
Paste or glue.
Chalk.
Paper.
Scotch tape for repairing tears in paper.
Staples for holding the papers that belong with one another.
Pens for marking on the overhead.
Towels for wiping all the markings off.
Band-Aids for covering up small wounds.
Cafeteria food.
Straws and napkins.
Gas or diesel fuel for bus rides.

How might these costs be calculated? Would multiplication or division be of any use?

Analysis...

The analysis of quantities and costs of items in a store can go on endlessly. All we need in hand is a calculator. All we need in mind is what it means to multiply or divide.

What does each cookie in a package cost to buy?
Of all the packages we analyze, which cookie costs the least?
Should we calculate by cookie size or weight?
Is the cheapest cookie the best to buy?
Should we factor in the taste?
Is soda in a king-size bottle cheaper for each serving than soda in a twelve-ounce can?
One sells for 49¢, two sell for 98¢. Is two instead of one a bargain we should rush to take?

The questions we can ask for cookies and for sodas we can ask for any item in the store.

Averages of all kinds...

The three kinds of averages we were taught in school were:

Mean: The arithmetic average of a collection of numbers.
Median: The middle number of a set of numbers placed in order from smallest to greatest or greatest to smallest.
Mode: The most frequent value or item in a set of data.

Averages are useful for comparing data of all kinds. Averages are most useful for helping to analyze the data from the graphs we have been making in our class.

We graph the birth months of every one in class and compare the most common birth month with the common months for all the other classes in our school. We use the mode to know the common months of birth. We can also graph the ages of all the students in our room. We use the mean to know the average age of every one in class. We find the mode by looking for the month or months that have the tallest columns on our graph. We find the mean by adding all the ages and dividing the total by the number of children represented by the graph.

Where else do numeric averages appear in our daily lives? Count the times a batter has been up to the plate. Divide the number of at-bats into the number of times the batter has hit the ball. The answer is a decimal number just like the decimal numbers found in the daily paper's sporting page. Averages of all kinds fill the sporting news. Averages for hitting, throwing, catching, shooting, winning, losing. Averages for almost anything that ends in "ing." How are these averages calculated anyway?

What kinds of averages are in the other sections of the daily paper? If today's temperature is warmer than the normal for the date, we use an average daily temperature to know what normal is. If rainfall for the year to date is three inches less than we should expect, then there is an average yearly rainfall against which this year's rain is being compared. How many years of rain were used to calculate this average? Which of the three kinds of average is the average in use? How many other kinds of averages are in the paper each day?

What kinds of averages do we use at school? Average daily attendance. Average number of students in attendance each year. Is the year-to-year population of our school going up or down? Average number of people eating in the cafeteria or riding the bus home from school. Average attendance at the monthly P. T. A. meeting. Averages for the team sports we play at P. E. Is there any limit to the averages we can find?

A written record...

We ask our students to find the cost of making a single sandwich or to calculate the gas mileage for the family car. We also ask our students to keep a written record of the problem-solving steps they use.

Mathematics is not a subject taught in isolation, just as language is not a subject we use only for an hour at the start of each school day. As our students find the cost of a peanut butter and jelly sandwich, they use more skills than the arithmetic operations they have learned. They think, they share, they research, they count and measure, they estimate and they decide.

Our students write down the ways they solve the problems that we give them so that we and they may have a written record of all the thinking involved. Writing is to be used to communicate our thoughts wherever there is thinking to be done.

We record our plans on paper as we make lists of what it is we will do today. We use written notes on our calendars to remind us what our future holds. We send our friends cards and letters with our written thoughts. We provide instructions of how to get from here to there by drawing little maps or using words to describe the path to take. We leave messages behind when the friend we stopped by to visit is not home. We record our thoughts on paper to keep track of what it is we think and feel. We use writing all the time.

We make writing a natural part of all our students do so that they can learn to write as frequently and as comfortably as we do. We know we have succeeded with our students when it occurs to them to write as often as we do.

Questions from Teachers

1. What is our assessment of division to be?

Assessment of the students or assessment of ourselves? We do both. When we were students in school, the assessment was the test. Because the test was the standard by which we were judged, the question asked for anything we learned was "Will this be on the test?" The difference between the daily assignments we were given and the tests was that the tests were timed. If we did not finish all the problems for the test, every problem left undone was simply counted wrong. Our ability to memorize rules and parrot back answers was the only measure of our skill.

Now, the ability to understand and reason is the measure of our students' skill. We do not measure reasoning by speed or by quantities of problems solved. The assessment of our students' thinking is contained within the quality of the problems that we give.

If we wish to find out if our students understand, we ask them to tell us what they know. We can ask them to answer individually or give us a response their work group decides upon.

Teacher: How would you explain division to a person who does not yet know how to divide?

Depending on our students' age and writing skills, we can ask for an oral response or a written one. If we wish to know more, we can ask:

Teacher: What kind of problems would you make up for this person to whom you had explained how to divide to test if that person really understood what you had said?

The first assessment question we ask as we assess ourselves is:

Have I decided what I want my students to know?

If we do not decide for ourselves what we want our students to know, someone else who does not know our students will make the decision in our stead. The price paid if we abandon the decision is:

A curriculum that does not meet the needs of every child in our room.
Methods of learning that convince children that they are terrible at math.
Lessons focused on teaching for the test and not teaching for our children's needs.

If we decide that we will decide what it is we want our students to know, we might decide:

I want my students to know how to divide.

There is no need to define our curriculum for division with any more precision than just one line. We know what we mean when we say "divide."

The second assessment question that we might ask ourselves is:

Am I creating an environment in my classroom that will ensure that each child learns?

We want our students to know how to divide. What does it take to ensure that each student learns?

Lessons that make sense to us as well as to our students.
Mathematics made a part of each student's life.
Help from every other student in the room.
Time for the learning to take place.

The next assessment questions that we might ask ourselves are:

Do I see the mathematics that is all around me all the time?
Do I use the mathematics that I see in the lessons that I teach?

Am I teaching for the child or am I teaching for the year-end test?

Assessment is a tool for finding out what we want to know—about our students and ourselves.

2. It seems that the more advanced the chapter, the fewer the problems we give the students to practice what they have learned. The *one-sandwich* problem could occupy our students for a week. Why do we seem to do fewer problems as we advance and not more?

If we teach addition, then subtraction, then multiplication, then division, each as separate skills, the number of problems of each kind we used for practice might remain the same. But we expect our students to use all the skills they already possess each time a new skill is introduced.

When we gave our students addition problems to contemplate, there was little else we asked our students to do but add. We had not yet provided our students with the background they need to be successful in using many more skills than that. As we teach each new skill, we select problem-solving questions that require continued use of the skills already learned. An adding problem has within it mostly adding. A division problem has within it mostly everything.

The one-sandwich problem has within it:

Addition	Algebra	Analysis	Application
Calculator skills	Classifying	Comparing	Contrasting
Data gathering	Decimals	Decision making	Estimation
Fractions	Geometry	Graphing	Logical thinking
Measuring	Multiplication	Parental involvement	Problem solving
Rounding	Sorting	Statistics	Subtraction
Time	Quantity		

...and, of course, division

As students know more, there is more to the math in each problem that we give.

In the city of Moscow, Russia, School 57 is a magnet high school for prospective mathematicians. Of the approximately 150,000 students eligible to attend the school each year, only twenty are selected.

The curriculum for the chosen ones consists of a handful of problems given at the beginning of each three- or four-week period. The students work the problems independently outside of class. They may consult with their teacher as often as they wish on the problem-solving methods. There is no specific time for problems to be completed. There are no grades assigned.

Of the 1,200 students who have passed through the program since its inception, 600 have entered the mechanical mathematics department at Moscow University and 300 have enrolled in the physical mathematics department. Nine have won International Mathematics Olympiads.

Fewer problems does not mean less math is being learned.