# Chapter 14

# Algebra

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## Before We Begin

**TIMSS...**

In response to a concern about our future economic well-being, our government has funded studies measuring mathematics and science learning internationally. These studies compare the academic achievement of children in the United States with other children in the world. The results have shown that among the forty-one countries studied, Japan ranks near the top, while the United States ranks significantly ahead of only seven other countries, none of whom are our economic competitors.

The Third International Mathematics and Science Study (TIMSS) added a new element to the information collected. In addition to testing student knowledge, extensive data was gathered on the teaching processes used in selected countries. In two countries in particular—Japan and the United States—class lessons at the eighth grade level were videotaped and analyzed, textbooks were studied and compared, and teaching standards were reviewed. The comparisons discussed below are for eighth grade classrooms in the two countries. The implications are for all the grades we teach.

**Findings...**

The most significant finding is that school matters—what is taught and how it is taught makes a real difference in the learning lives of students. Schooling is the prime explanation of the differences in results—not culture, not time in class, not hours of homework. U. S. students spend more time studying mathematics and more hours doing math homework than do their counterparts in Japan.

Since what is taught and how it is taught is the most significant factor in the differences in learning, and since the TIMSS collected data on the teaching processes used, we can learn from the TIMSS results what makes a difference.

Among the areas analyzed, the following were compared:

Goal for the lesson
The lesson
How class time is spent
Textbooks
Assessment
Homework
Teacher preparation
Concepts presented in the lesson
Use of the overhead projector
Topics covered
Tracking
Outside interruptions
Kinds of content at eighth grade
National Standards

The comparisons showed:

**Goal for the lesson**

73% of Japanese teachers asked to state a primary purpose for a lesson said having students understand a mathematical concept was the goal.

61% of U. S. teachers asked the same question said the correct performance of a particular skill was the goal.

**Concepts presented in the lesson**

In a Japanese lesson, the concept to be taught is developed at the beginning of the lesson, with the focus on understanding. Students actively participate in the concept's development.

In a typical U. S. lesson, the concept is stated, not developed. No understanding of the concept is expected. The concept is learned through drill.

**The lesson**

In Japan, math lessons are coherent packages, focused on one mathematical concept. A representative Japanese lesson begins with the statement of a problem rich in mathematical content. Students struggle with the problem individually, then work in small groups to share and expand their proposed solutions. As the students work, the teacher walks around observing and commenting on the methods being employed. Creativity and imagination are encouraged. Solutions developed by the small groups are then presented to the whole class for discussion. Varied, but equally acceptable versions of ways to solve the problem are presented. At the lesson's end, the teacher summarizes the concepts developed for that day. The lesson balances practice and thinking. The ideas from one lesson are connected to ideas presented earlier.

A representative U. S. lesson begins with an explanation from the teacher on how to perform a skill. Problems that practice the skill are then assigned. Students work on the assignment as the teacher walks around tutoring individuals. In a TIMSS assessment of the mathematical content of the lessons presented, no lesson in any of the U. S. schools was found to have high math content. A full 93% of U. S. lessons reviewed were rated low or disconnected in terms of content.

**Use of the overhead projector**

Although overhead projectors were available, Japanese teachers almost never used them, opting for chalkboards instead.

U. S. teachers used overhead projectors for presentations about half the time.

Reason:

Japanese teachers operate on the assumption that all children's minds wander. Lessons written on the chalkboard can be left unerased for extended periods of time. A child who tunes out then back in has a written record on the board of what has happened in his or her mental absence. In addition, a written record of the lesson left in place leaves a visual trail for the slower child to ponder at his or her own pace.

U. S. teachers use the overhead to guide the students' attention. As new problems are presented, old problems are erased. There is generally no visual trail of the lesson taking place.

**How class time is spent**

Japanese students spend the majority of their class time working on inventing solutions, solving problems and proving their findings.

U. S. students spend the majority of their time in drill, or in reviewing and doing homework.

**Topics covered**

Japanese teachers cover substantially less mathematical topics in a school year than their U. S. counterparts. Topics are presented with an emphasis on a balance of practice and thinking.

U. S. teachers attempt to cover over thirty topics a year. The emphasis on covering so many topics leads to a curriculum that has been described by the TIMSS as a mile wide and an inch deep.

**Textbooks**

Japanese textbooks are small and inexpensive. The topics presented are covered in depth.

U. S. textbooks are the biggest and most expensive in the world, with more topics presented than in any textbook of any of the countries that perform at higher levels than the U. S. Money spent on large textbooks in the U. S. is available in Japan and other countries to be spent for teacher training and student materials.

**Tracking**

There is no tracking in Japanese schools. Every student is presented the same content.

U. S. middle schools track students by supposed mathematical ability, assigning most to review past concepts, some to pre-algebra and some to algebra. Not all students receive the same content.

**Assessment**

Testing measures made in Japan are used to help teachers judge the effectiveness of their teaching efforts so they may devise improvements in their lessons. No standardized tests are used to compare schools or judge teachers.

Use of standardized tests in the U. S. further fragments the curriculum, since the multiple choice aspect of the tests measures only the most basic skills, and the curriculum reflects what is to be tested. The tests do not have as their purpose assisting teachers in modifying their lessons.

**Outside interruptions**

In none of the lessons observed for the TIMSS in Japanese classrooms were there any interruptions to the math lessons. Teaching time is as uninterruptable as we might regard a church service.

In 31% of the lessons observed for the TIMSS in U. S. classrooms there were interruptions to the math lessons. Interruptions included such things as general announcements over the school intercom, outside messengers collecting attendance sheets, and visitors of one kind or another.

**Homework**

In Japanese lessons, homework does not play a significant part in mathematics learning.

Many U. S. classes begin the period discussing the previous night's homework, and much class time is spent at the end of the lesson beginning work on that night's homework assignment.

**Kinds of content at eighth grade**

In Japan all eighth grade students study algebra and geometry.

In the U. S. some algebra is taught but generally only to students tracked as capable of learning algebraic skills. Geometry is not taught.

**Teacher training**

The problems that are the focus of the Japanese mathematics lessons are formulated by groups of teachers who create the problem and plan the lesson together. Once a problem has been agreed upon, one of the group teaches it to a class of students while the other group members observe. After the lesson, the group critiques and refines the problem to answer the question, "How can we improve the lesson?" Japanese teachers have more prep time than their U. S. counterparts and spend less hours in class.

U. S. teachers do not receive as much practical training and daily support as their Japanese counterparts. They spend more time in front of their students with less time to prepare. There is little teacher involvement in creating curriculum.

**National Standards**

In Japan the same basic plan for education has been in place for fifty years. Fifty years of steadily improving the lessons taught and the questions asked. There is no concept of reform in education, only of improvement. There is no swinging pendulum.

Mathematics education in the U. S. is uneven and disjointed. There are 15,000 autonomous school districts and, in many cases, school districts may set their own educational standards. Also setting standards or guidelines for teaching are: State boards, textbook publishers, test makers, professional organizations, and parent groups. No single coherent vision guides mathematics instruction. The vision is a splintered one—a Tower of Babel.

For further information on the TIMSS see *Splintered Vision: An Investigation of U.S. Mathematics and Science Education, Executive Summary*, National Library of Education, Washington D.C.

**The message...**

The message of the TIMSS is that teaching matters. How we teach relates directly to what our students learn. The educational well-being of the next generations is in our hands. *Everybody Counts* (page 001) informs us that communication has created a world economy in which working smarter is more important than merely working harder. Jobs that contribute to this world economy require workers who are prepared to absorb new ideas, to adapt to change, to cope with ambiguity, to perceive patterns, and to solve unconventional problems. It is these needs, not just the need for calculation, that make mathematics a prerequisite to so many jobs. The message of the TIMSS is that if we want our children to be able to think, then thinking must be what we teach.

Just as finding more than one path to reach an answer is encouraged in a Japanese classroom, there is more than one path we might use to reach the goal of having our students understand the math they learn. The message of the TIMSS is not the path traveled, it is the destination reached.

The message of the TIMSS is also that countries in the world with which we compete economically teach algebra to all their middle grade students, while we teach algebra only to a chosen few.

**Algebra is...**

Algebra is the branch of mathematics in which letters and other symbols represent basic arithmetic relationships. It is the graphing of functions and relationships in quantity, chance, shape, rates of change, statistics and geometry. The data that we graph most often comes from situations in which the variables are related in some patterned way. Algebra is the language that mathematicians use to record patterns and connections. It is how we record the patterns that our work with manipulatives discovers. It is how we write the rules for the patterns that we see. It is an attitude, an awareness, a state of mind. It is a way of thinking.

Algebra is so central to mathematics that students who do not learn algebra severely limit their future options. Without algebra, students are cut off from future mathematics study and from advanced careers in business, technology and the sciences. Algebra is the gateway to more advanced academic and vocational opportunities.

**John's story...**

John, a high school student, talking to his uncle:

**John: I am really doing well in algebra this year.**
**Uncle: That's nice to hear. What do you mean when you say you are doing well?**
**John: I'm getting all As. I will be in the advanced-placement class next year.**
**Uncle: What do you use algebra for?**
**John: What do you mean?**
**Uncle: Can you give me an example of a situation where you would use algebra to solve a problem you might find during the day?**
**John: Algebra doesn't have any real uses. Its just a subject that we learn in school. The only use it has is for doing well on college entrance tests.**

John learned algebra in high school, but what did he learn?

**The story told again...**

Is John's story unique? At the N.C.T.M. national conference for teachers of mathematics, a textbook publisher brought in a group of high school students to help promote its algebra series. The program was not the same one John had used, but it might as well have been. The publisher had given all the students T-shirts to wear that said "I love (name of program) math." When not staffing the publisher's display booth, groups of the T-shirt clad youths roamed the conference exhibition floor vocally promoting the algebra program they were brought there to extol.

Five or so boys and girls descended upon a neighboring exhibit booth chanting "We love (name of program) math!" A teacher also visiting this particular booth engaged the youths in conversation:

**Teacher: Did any of you learn algebra this year?**
**Youths: We all did.**
**Teacher: How did you do?**
**Youths: We did great!**
**Teacher: What do you mean by great?**
**Youth: We all got As and Bs and we will all be in the advanced-placement class next year.**
**Teacher: Can any of you tell me what you use algebra for?**
**Youth: You mean like write out some equations and solve them?**
**Teacher: No. I mean, can you give me an example of when you would use algebra to solve any math problem except a problem in your book?**
**Youth: Our algebra class got the highest scores in our school on the placement test using (name of program) math.**
**Teacher: Yes, I'm sure you did. Your program is famous for having everybody do well on tests. But what did your program teach you about any use algebra might have?**
**Youth: We didn't learn things like that. We just learned algebra. Algebra is important for getting into college. It doesn't have any uses except for that.**
**Second youth: Maybe engineers use it, but I don't want to be an engineer.**
**Teacher: Okay. Let's borrow a sheet of graph paper from this booth.**
**Lets take one of these circular objects from this booth and mark its diameter along the x-axis and its circumference along the y-axis.**

(illustration 14-0-1)
(The graph paper has an x and y axis drawn on it. The roll of tape is used as an example of how to mark the diameter along the x-axis and the circumference along the y axis. For the diameter, one edge of the roll is placed at the (0,0) point on the graph and the opposite edge is marked along the x-axis. For circumference, a small pencil mark is made on the outside edge of the tape. The roll is placed at the (0,0) point on the graph, with the pencil mark at 0,0. The tape is then rolled carefully along the y-axis until the pencil mark has made a full revolution. A mark is made on the y-axis at the point where the full revolution is completed. A light line is drawn up from the diameter measurement on the x-axis and a light line is drawn over from the circumference measurement on the y-axis. Where these two lines meet is the single coordinate point for the circumference and diameter measurements.)

**Teacher: Okay, we've plotted the circumference and diameter coordinates for one circular object. Now, let's plot more coordinates for some of these other circular objects and connect the points.**

(illustration 14-0-2)
(Coordinate graph with several points plotted. Label each point with the name of the object the point represents. Tape, coin, lid, etc. Connect the points with a line that extends through and beyond them all.)

**Teacher: How do you find slope?**
**Youth: Divide the rise by the run.**
**Teacher: Divide vertical distance by horizontal distance?**
**Youth: Yes.**
**Teacher: What is the slope of this line? Just estimate it.**
**Youth: Well, it looks like when the diameter is one, the circumference is a little over three.**
**Teacher: You mean when the run is one, the rise is a little over three?**
**Youth: Yes.**
**Teacher: Pretty good estimate. If we measured really carefully, I think we'd find the slope is something like 3.14159.**
**Youth: Pi?**

**Teacher: Yes, the slope of this line is Pi. If we had the time and the inclination to graph diameters and circumferences for some more circles, do you think their points would be on this line you've drawn, assuming you extended the line out far enough?**
**Youth: I guess they would. All circles would probably be on this line.**
**Teacher: What's the equation for the line we've drawn?**
**Youth: $y = \pi x$**
**Teacher: Or, if we substitute c (circumference) for y, and d (diameter) for x, the formula would be $c = \pi d$, the formula for finding the circumference of a circle if you know the diameter. Have you ever done anything like this in your algebra class?**
**Youth: That's not in our book.**
**Teacher: Well, that's the concern people have with the program you say is so great. All your program has you do is memorize the rules. A calculator is good for finding answers, but it cannot find a single answer if its buttons are not pushed. What good is the program you all say you love so much if you haven't learned a single use for any of the algebra that you know, except for passing tests? All you've learned is to wait until your buttons have been pushed.**

The youths walked away from the exhibit booth and did not return.

**Grade-school algebra...**

Algebra may be important for our students to learn, but why concern ourselves with algebra in the elementary grades? We concern ourselves because other countries do. Our students will be earning their livings in a global market place. Their academic preparation for the world begins in our classrooms. It is within our power to send our students on to higher grades with an understanding of algebra just as other countries do.

As an example of what other countries teach, the English National Curriculum Guide lists these attainment targets for grade-school algebra:

Recognize and use patterns and relationships & sequences and make generalizations.
Copy, continue and devise repeating patterns represented by objects or one-digit numbers: Continue the pattern red, red, blue, red, red, blue. Continue the pattern 2, 1, 2, 1, 2, 1, 2, 1.
Explore and use the patterns in addition and subtraction facts to 10: Use counters to make various combinations for a given total.
Distinguish between odd and even numbers.
Explain number patterns and predict subsequent numbers where appropriate: 5, 10, 15, 20, etc. 2, 4, 6, 8, 10, etc.
Apply strategies, such as doubling and halving to explore properties of numbers, including equivalence of fractions: $23 \times 8 = 46 \times 4 = 92 \times 2 = 184 \times 1$.
Generalize, mainly in words, patterns that arise in various situations.
Determine possible rules for generating a sequence. Use symbolic notation to express the rules.
Use spreadsheets or other computer facilities to explore number patterns.
Calculate growth rates and display graphically.
Understand the use of a symbol for an unknown number: $3 \times [\ \ ] = 10$.
Understand the use of function machines: input--> machine--> output.
Understand and use simple formulas or equations expressed in words or symbols.
Solve linear equations.
Plot points on a coordinate graphs.
Understand and use coordinates in all four quadrants—positive and negative.
Understand uses of negative numbers.
Use coordinate graphs to solve problems and record data.
Generate various types of graphs on a computer or calculator and interpret them.
Recognize and use functions, formulas, equations and inequalities.
Use graphical representations of algebraic functions.
Use algebra in practical tasks and real life situations.
Use algebra to pursue investigations within math itself.

From: *Mathematics in the National Curriculum*, Department of Education and Science, London, England, 1989.

In short, the attainment targets ask English students in the elementary grades to:

Look for patterns.
Use patterns found to predict and generalize.
Record patterns in words and symbols.
Create patterns of their own.

Understand the connection between patterns and formulas (formulas are patterns generalized).
Use coordinate graphs to record data to examine for patterns.
Understand the use of functions, with and without function machines.
Understand the use of negative numbers.
Know that these skills are collectively called algebra.
Use algebra in real life.

As we read the list, we can see the kinds of teaching that connect to algebra flow naturally from the mathematics we have been teaching all along. From A-A-B patterns through finding patterns for areas on a geoboard; from keeping track of rates of growth to finding equivalencies; we have already begun introducing algebra to our students, regardless of the grade we teach. Patterns are everywhere. Algebra is everywhere that patterns are.

**What else might we do?...**

If the kinds of teaching that connect to algebra flow naturally from the mathematics we have been teaching all along, what else might we do to insure our students' future success in algebra? We can prepare our intermediate students for the algebra they will face in middle school and high school. We can provide specific preparation to make more clear the connection between elementary mathematics and the formal courses in algebra that await our students in the higher grades.

The ten lessons that follow present specific forms of algebra that our students will encounter in middle schools. The topics covered are:

1. Patterns, relationships and formulas.
2. Learning how to use positive and negative coordinates.
3. Negative numbers, including multiplying and dividing signed numbers.
4. Functions and variables.
5. Equations with graphing.
6. Studying equations and graphs
7. Exponents and roots (radicals)—more equations to explore and graph.
8. Solving equations—(x+y)(x+y).
9. Ratios, rates, proportionality and equivalencies.
10. Word problems of all kinds—algebra and coordinate graphing in real life.

In the higher grades, the object of the lessons will be learning how to apply algebraic rules. In the elementary grades, the point of the lessons is the exploration, not the rules. Since there is nothing in particular our students are required to know of algebra, there is no requirement to produce a particular result. This means our students have the maximum opportunity to think creatively.

**Lesson One**

| | |
|---|---|
| Purpose | Learn to write symbolic equations or formulas for familiar patterns or experiences. |
| Summary | Students look at geoboard, Power Block, and wooden cube patterns and use letters to record the patterns seen. |
| Materials | Power Blocks, geoboards, wooden cubes. |
| Topic | Squares and rectangles made with S-1 squares, lengths, widths and areas recorded and searched for patterns. |
| Topic | Boxes made with wooden cubes, with the formula for volume sought. |
| Topic | Geoboard formulas for area and Pick's theorem. |
| Topic | Algebraic relationships between the Power Blocks. |
| Homework | Pattern searches leading to Pick's Theorem are continued at home for everyone to share. |

**Patterns, relationships, and formulas...**

Algebra is the branch of mathematics in which symbols are used to represent numbers. Letters represent basic arithmetic relationships.

**Arithmetic**

**Teacher: Name a number.**
**Student: Seven.**
**Teacher: Name a number that is one greater.**

**Student: Eight.**

**Algebra**

**Teacher: Name a number.**
**Student: n.**
**Teacher: Name a number that is one greater.**
**Student: n + 1.**

Lesson One asks our students to look at mathematical patterns and experiences already familiar to them and write symbolic equations or formulas for the patterns seen. Equations are how mathematicians write the rules for patterns. Formulas are written records of patterns seen.

**Area...**

**Teacher: How many different sizes of squares and rectangles can we make using 24 S-1 squares, with no squares left over? Make a table of lengths and widths for all the areas of twenty-four that you find.**

| Length | Width | Area |
|---|---|---|
| 1 | 24 | 24 |
| 2 | 12 | 24 |
| 3 | 8 | 24 |
| 4 | 6 | 24 |
| 6 | 4 | 24 |
| 8 | 3 | 24 |
| 12 | 2 | 24 |
| 24 | 1 | 24 |

**Teacher: Is there a pattern you can see in the lengths and widths that might tell you if we have found all the lengths and widths we can for an area of 24?**

The pattern is that area is found by multiplying length times width. This is a pattern our students found when using squares in learning how to multiply. In algebra we ask our students to write equations for the patterns seen.

**Teacher: In algebra, we would record the pattern as:**

$$lw = a$$

**Letters written next to each other mean multiply. What do you think the letters in the formula that I have written mean?**
**Student: Length times width equals area.**
**Teacher: Is this same formula true for areas of other rectangles or squares?**

Other questions we might ask:

Would the pattern be the same for all the different sizes of squares and rectangles that could be made using 36 squares, with no squares left over?
Is there any square or rectangle for which the formula will not work?
Could we use the formula to help us know when we have made all the different rectangles that can be made for a given number of squares?

What other questions might we think to ask about squares or rectangles or our formula? We have asked our students to search for patterns countless times before. With every search, we were laying a foundation for algebra. Algebra is the written record of patterns found.

**Boxes with wooden cubes...**

**Teacher: What kinds of box-like shapes can you make with wooden cubes?**

(illustration 14-1-1)
(Box-like shapes constructed from stacks of wooden cubes. Long ones, tall ones, cubic ones.)

**Teacher: Make a table of lengths, widths and heights for the boxes that you make. Record the number of cubes you use as well.**

(illustration 14-1-2)

(Chart with Boxes written at the top. Four columns: Length, Width, Height, Volume. Numbers for various boxes filled in.)

**Teacher: What patterns do you see?**

If our students have seen that length times width equals area, will they see the pattern for volume as well? We can use our questions as a guide.

**Teacher: The pattern that you saw for the area of rectangles was length times width. I wonder if there is a pattern for length times width times height that would tell you the volume of the boxes you have made?**
**If there is, then could you use this pattern to help you know the number of cubes you will need to make a box of a particular size before you build it?**

If our students discover that length times width times height equals volume, we show them how mathematicians record this pattern.

**Teacher: Here is how we record the pattern algebraically.**

$$lwh = v$$

**Is there a particular number of cubes that will give you the greatest number of different box-like shapes? Can you make more box-shapes with, say, 36 cubes than you can with 45?**
**Could we use the algebraic formula to tell us when we had made all the different box shapes from a particular number of cubes?**

What other questions might we think to ask about the boxes that our students make or the formula that is the algebraic expression of the pattern found?

**Geoboards and Mr. Pick...**

For geoboards, we said, "Let's see if we can find a pattern in the areas of the triangles you have made that might help us predict the areas for triangles we have not yet made." (*Fractions*, Lesson Two, page 000.) If our students found that the pattern was: "Area equals half the base times the height," the pattern can be recorded algebraically as:

$$a = 1/2bh$$

The formula for squares and rectangles on a geoboard is the same as the formula our students found using the S-1 squares.

(illustration 14-1-3)
(Show a table of heights, bases and areas for rectangles, with the lw = a formula written underneath.)

What might be the formula for parallelograms?

(illustration 14-1-4)
(Define "height" and "base." Show a table of heights, bases and areas for parallelograms, with the formula written underneath.)

What might be the formula for trapezoids?

(illustration 14-1-5)
(Define "height, "base-1" and "base-2". Show a table of heights, base-1s, base-2s and areas for parallelograms, with the formula written underneath. Explain why it is necessary to record bases 1 and 2.)

For each new shape, there is a pattern our students may discover and record algebraically. Once our students discover the patterns for the shapes individually, there is yet another pattern to explore. A gentleman named Pick says there is a single pattern for the areas on geoboards that transcends the names we might give the individual shapes.

**Teacher: You found patterns in the bases and heights of triangles, rectangles and trapezoids that let you predict their areas. Each of these shapes has a pattern of its own. Can you find a single pattern for predicting the area of any shape?**
**Make shapes on your geoboard for which you can find areas. Then for each shape, record separately the area, the boundary nails and the nails inside the shape. Here is what I mean.**

(illustration 14-1-6)
(Show several geoboard shapes with the area, the boundary nails (the nails touched by the rubber band) and the inside nails (the nails not touched by the rubber band) recorded beneath the shape. Then show a table in which all the data from the shapes is recorded. Label the table's columns a (for area), b (for boundary nails) and i (for internal nails).)

**Teacher: What pattern can you find for the boundary and inside nails of all these shapes that might let you predict the areas?**

The pattern, called Pick's Theorem, is represented by the formula: $a = b/2 + i - 1$. Will our students find this pattern? Does it matter if they do? Or, is the search more important for their thinking than any answer they might find?

We can increase the likelihood of our students' discovering the truth of Pick's theorem by sending the pattern search home to share with family members and anyone else in the neighborhood. Mathematics learning is not confined to the children in our room.

**Power Blocks...**

For Power Blocks, we said, "The S-1 square has an area of one. Find the areas of the other shapes. Keep a record of your proofs. Now, the T-1 triangle has an area of one. Find the areas of the other shapes. Keep a record of your proofs. Now, the S-5 square has an area of one..." (*Fractions*, Lesson One, page 000.) As we change the shape that we called "one", the values of every other shape change, as well. We can also ask our students to use the letters on each Power Block to write algebraic equations for the relationships.

(illustration 14-1-7)
(Show the S-5 square with T-5 and 2 T-4's next to it formed into the shape of the S-5. Write the equation S-5 = T-5 + 2 (T-4) beneath the shapes.)

The value of the shapes may change as we change the shape that we call one, but the algebraic equations for their relationships remain the same.

**The difference is...**

S-1 squares, wooden cubes, geoboards, Power Blocks—materials our students use to understand the concepts that underlie arithmetic. Now our students use the same materials to understand the concepts that underlie algebra. The difference between arithmetic and algebra is not the concepts. The difference is the recording used. Arithmetic uses numbers to record specific concepts—like $2 + 2 = 4$. Algebra uses symbols to record broader concepts of which the specifics are a part—like $x + y = 4$.

As we gain experience in seeing the algebra that is already present in the arithmetic we teach, the number of connections that we make between arithmetic and algebra will grow. Any time we see a pattern that lets us anticipate what the next numbers in a sequence are to be, we can use algebra to record the pattern seen. The formulas or equations that we record give us the power of knowing the 100th number in a sequence without having to calculate individually the ninety-nine that go before.

## Lesson Two

| | |
|---|---|
| Purpose | Learn to plot coordinate points on a graph. |
| Summary | We play coordinate tic-tac-toe with our students as a class. Students play the game of Battleships with themselves. |
| Materials | Graph paper. |
| Topic | Students play coordinate tic-tac-toe as a class. |
| Topic | Students play Battleships in small groups. |
| Homework | Students may continue playing Battleships at home. |

**Descartes' coordinates...**

In his efforts to systemize geometry, René Descartes (1596-1650) classified geometric curves by the types of equations that produced them and, in the process, created the coordinate graphing that is an elemental part of algebra. Descartes' graphing of equations converted recorded patterns into visual pictures of all the pattern's possibilities. Descartes' coordinates changed recording from knowing what is to knowing what will be.

In Lesson Two our students begin the process of learning how to graph coordinately, so that by Lesson Five they will have learned to convert equations from symbolic written records to lines that show all the possibilities. Algebra uses letters to represent relationships. Letters may be used, and so may lines.

**Tic-tac-toe and Battleships...**

A process for introducing our students to the coordinates of coordinate graphing is already contained in *Mathematics... a Way of Thinking*. Lesson 18-5 introduces the game of Coordinate Tic-Tac-Toe. Lesson 19-4 adds negative numbers to the game.

The game of Battleships teaches the use of coordinates just as well. For those of us who did not pass our youthful time playing Battleships on rainy days and whose own children never had a store-bought version of the game, the rules are included here.

Materials: Pencils and a few pieces of graph paper.

The game: One child or team of children play against another child or team. Each team has six ships to place upon its graph paper sea. The number of ships is arbitrary, as numbers in most games are. The playing area is a section of graph paper with the coordinates 0 through 10 marked off on the x and y axes. The playing area is equally arbitrary. Larger oceans make for longer games. For the example here, the six ships and their sizes are:

1 Air craft carrier..........................5 connected squares in a line
2 Battleships................................4 connected squares each, in a line
2 Destroyers.................................3 connected squares each, in a line
1 Submarine ................................2 connected squares in a line

(illustration 14-2-1)
(Six ships placed on a sheet of graph paper with coordinates marked from 0 to 10 on the x axis and from 0 to 10 on the y axis. The 0 on each axis is the same point.)

Ships are placed horizontally or vertically. If both teams of players agree, the ships may also be placed on 45° diagonals. Whichever direction the ships are placed, each ship must have all its squares in a straight line. Neither player or team lets the other player or team see where its ships are placed.

Once the ships are placed, each team takes turns firing shots or lofting bombs at the other side's ships by saying the coordinates of the square to which the shot or bomb is directed. The numbers must be said x, or horizontal axis, first and y, or vertical axis, second.

The players or teams take turns firing at each other's ships until all the ships in one ocean or the other are sunk. If a ship is hit, the player with the hit ship must say "hit." The player does not have to identify the kind of ship hit until it is actually sunk. It is up to the firing team to shoot or bomb the squares around the hit ship until the player with the hit ship says "sunk." The player or team scoring a hit gets another turn after each hit made.

**First team: 2, 3?**
**Second team: Miss. 4, 5?**
**First team : Miss. 6, 7?**
**Second team: Miss. 2, 8?**
**First team: Miss. 8, 9?**
**Second team: Hit.**
**First team: 8, 8?**

(illustration 14-2-2)
(Show a game of Battleships played part way through. Label the two different sheets of graph paper 'First team' and 'Second team'. Record the moves from the dialog above on the sheets.)

Players often use a second sheet of graph paper to record all the shots taken at the other side, so the same squares are not bombed twice, and so there is a record of the hits on the other player or team's ships. The teacher watches students who are playing for the first time to see that they know to match the numbers given for each shot with the proper x, y coordinates.

As is true for Coordinate Tic-Tac-Toe, once students understand recording moves in the positive quadrant, negative axes are added to the game.

(illustration 14-2-3)
(Show a four quadrant coordinate graph. In each quadrant show a point with its coordinates written alongside.)

The parallel points of both Coordinate Tic-Tac-Toe and Battleships are to teach our students that two numbers describe a point, and to teach our students that the commutative rule does not apply to coordinates—the point (2, 3) is not the same as the point (3, 2).

**Lesson Three**

| | |
|---|---|
| Purpose | Learn a framework for understanding addition, subtraction and multiplication of signed numbers. |
| Summary | We present Letter Carrier stories to our students to teach them rules for arithmetic operations with positive and negative numbers. |
| Materials | Chalk boards. |
| Topic | Letter Carrier stories for + and —. |
| Topic | Letter Carrier stories for x. |
| Topic | Students create stories for numbers we provide. |

**Expanding possibilities...**

Plotting negative numbers on a coordinate graph is no more difficult that plotting numbers that are positive. Whether a number is negative or positive, the first number in each coordinate pair is the x axis position, the second number is the y axis position. There is more to understanding negative numbers, though, than knowing where to mark a number on a graph.

In the early grades, our students learned that 9 - 6 = 3, but 6 - 9 was a problem they could not solve. With the introduction of negative numbers, 6 - 9 becomes solvable: $6 - 9 = {}^{-}3$. Negative numbers expand the possibilities.

If we ask our students to record all the ways to make 10 with two groups of S-1 squares, they might list 1 + 9, 2 + 8, 3 + 7, 4 + 6, and 5 + 5 as all the possibilities. All the ways can be recorded algebraically as: $x + y = 10$. The graph for this equation includes more ways than we can represent with S-1 squares.

(illustration 14-3-1)
(Graph of x + y = 10. Mark the points (2 1/2, 7 1/2) and (11, $^{-}1$) on the graph.)

The graph above includes 2 1/2 + 7 1/2 as a way to make 10. How many other fractional or decimal points are there on the line? The graph also includes $11 + {}^{-}1$ as a possibility. Is $11 + {}^{-}1$ the same as 11 — 1? No and yes. To know why negative numbers are and are not like numbers we subtract, our students need a broader understanding of negative.

**The letter carrier...**

A framework for understanding addition and subtraction of negative numbers is contained in Chapter 19 of *Mathematics... a Way of Thinking*. Chapter 19 introduces students to a letter carrier who is more confused than letter carriers ought to be. This letter carrier brings mail containing checks and bills to our home, but the checks and bills are not necessarily addressed to us. So, the letter carrier must sometimes take back what has been delivered to correctly deliver it somewhere else.

When the letter carrier brings a check for \$5, we are richer by 5. A check is a positive and is recorded as $^{+}5$. Bringing a check is recorded as $+ {}^{+}5$.

When the letter carrier brings a bill for \$5, we are poorer by 5. A bill is a negative and is recorded as $^{-}5$. Bringing a bill is recorded as $+ {}^{-}5$.

When the letter carrier needs to take back a misdelivered check, we are poorer, because money we thought we had is no longer ours to spend. Taking back a check is recorded as $— {}^{+}5$.

When the letter carrier needs to take back a misdelivered bill, we are richer, because a debt we thought we owed is no longer ours to pay. Taking back a bill is recorded as $— {}^{-}5$.

Once we have introduced the letter carrier framework, we pose problems and guide our students in writing the numerical expressions that the problems represent. Our students write the numbers and the signs and tell us if the letter carrier has left us better or worse off. We provide as much or as little help as our students need in recording the actions that our problems pose.

**Teacher: The letter carrier brings us a check for $6 and a bill for $4. Write the equation for what we received on your chalk boards and tell me how much better or worse off we are.**

**Students: $^{+}6 + {}^{-}4 = {}^{+}2$. We are better off by $2.**

**Teacher: The letter carrier brings a bill for $3 and then brings another bill for $4. Write the equation for what we received and tell me how much better or worse off we are.**

**Students: ${}^{-}3 + {}^{-}4 = {}^{-}7$. We are worse off by $7.**

**Teacher: The letter carrier brings a check for $5 and takes away a bill for three. Write the equation and tell me how much better or worse off we are.**

**Students: ${}^{+}5 — {}^{-}3 = {}^{+}8$. We are better off by $8.**

**Teacher: The letter carrier brings a check for $4 and takes away a check for $6.**

**Students: ${}^{+}4 — {}^{+}6 = {}^{-}2$. We are worse off by $2.**

**Teacher: The letter carrier takes away a bill for $4 and then takes away a bill for $5.**

**Students: $— {}^{-}4 — {}^{-}5 = {}^{+}9$. Better off by $9.**

**Patterns in the signs…**

As our students work, we ask them to look for patterns in the individual actions of the letter carrier. For each check or bill received or taken back, when are we better off and when are we worse off?

A bill for $5 that is taken away is recorded — ${}^{-}5$, and leaves us better off by 5. A negative subtracted is a positive.

A check for $5 brought is recorded + ${}^{+}5$, and leaves us better off by 5. A positive added is a positive.

A bill for $5 that is brought is recorded + ${}^{-}5$ and leaves us worse off by 5. A check for $5 taken away is recorded — +5, and leaves us worse off by five. A negative added and a positive subtracted are both negative.

The rules for working with positive and negative numbers that our students can discover are:

| | | | |
|---|---|---|---|
| + | $^{+}$ | --> | + |
| — | $^{-}$ | --> | + |
| + | $^{-}$ | --> | - |
| — | $^{+}$ | --> | - |

**Multiply…**

The letter carrier stories give us a framework for understanding adding and subtracting signed numbers. What happens when we multiply?

**Teacher: The letter carrier brings us three bills for $5. Write the equation for what we received and tell me how much better or worse off we are for what we have received.**

**Students: ${}^{-}5 + {}^{-}5 + {}^{-}5 = {}^{-}15$. We are worse off by $15.**

**Teacher: Three bills for $5 can also be written as $3({}^{-}5) = {}^{-}15$. In algebra, a number written outside the parenthesis means the numbers inside are to be multiplied by it.**

**New problem. The letter carrier takes four bills for $6 away. In this case, the four bills taken away would be written as $—4({}^{-}6)$. Write the equation and tell me if we are better or worse off.**

**Students: $—4({}^{-}6) = {}^{+}24$. We are better off by $24.**

**Teacher: The letter carrier brings three bills for $5 and takes four bills for $6 away. Write the equation and tell me if we are better or worse off.**

**Students: $3({}^{-}5) —4({}^{-}6) = {}^{+}9$**

It is not likely that our students will know initially how to write the problems that we pose. It may also be that we ourselves are not always sure how the numbers and the signs are to be written down. Is the last problem posed above to be written $3({}^{-}5) —4({}^{-}6)$, or is it written as $3({}^{-}5)$ **+** $-4({}^{-}6)$? Would it make any difference which way we chose? What framework are we, as teachers, to use to guide the learning that our students do, when the algebra that we learned may be a distant part of a forgotten past?

To know the numbers and the signs to use, we must do what we expect our students to do—we must think. We use the same framework of bills and checks delivered and taken away that our students use to determine what is reasonable. What do the numbers in this problem mean, and what might the answer be?

$$+4(^{-}7) — 5(^{-}3) = {}^{-}28 + 15 = {}^{-}13$$

The first sign in any set of numbers tells us what happened. The second sign tells us the thing it happened to. The + before the 4 tells us that the letter carrier brought four of something to our house. The - in front of the 7 tells us that what was brought was bills for $7. The arrival of 4 bills for $7 leaves us worse off by $^{-}28$. The — before the 5 tells us that the letter carrier took five things away. The - in front of the 3 tells us that bills for $3 were taken back. The removal of 5 bills for $3 leaves us better off by $^{+}15$. If we combine a worse off $^{-}28$ with a better off $^{+}15$, we are still worse off by $^{-}13$.

$$+4(^{-}7) — 5(^{-}3) = {}^{-}13$$

**More patterns in the signs...**

We asked our students to look for patterns in adding and subtracting. We ask for patterns in the times, as well. For each multiple check or bill received or taken back, when are we better or worse off?

Several bills (b) for $5 each that are taken away are recorded —(b)($^{-}5$), and leave us better off by several times the 5. A negative times a negative is a positive.

Several checks (c) for $5 each that are brought are recorded +(c)($^{+}5$), and leave us better off by several times the 5. A positive times a positive is a positive.

Several bills (b) for $5 that are brought are recorded +(b)($^{-}5$), and leave us worse off by several times the 5. Several checks (c) for $5 that are taken away are recorded —(c)($^{+}5$), and leave us worse off by several times the five. A negative times a positive and a positive times a negative are each negative.

The rules for multiplying positive and negative numbers that our students can discover are:

| | | | |
|---|---|---|---|
| + | + | --> | + |
| — | - | --> | + |
| + | - | --> | - |
| — | + | --> | - |

**Clarity...**

Parenthesis are used for clarity. Parenthesis are also used to tell us which operations to perform first. If the letter carrier brought five bills for $4 and five checks for $3, we write the bills and checks as:

$$+5(^{-}4) + 5(^{+}3) = {}^{-}20 + {}^{+}15 = {}^{-}5$$

Or, we could write it as:

$$+5(^{-}4 + 3) = +5(^{-}1) = {}^{-}5$$

The rule for parenthesis is that the arithmetic inside is done before the arithmetic outside.

The question for our students to reason out for each event is: Are we better or worse off financially after the letter carrier's delivery or misdelivery? Our students provide the thinking, with our help. Positive and negative numbers are the signs used to record the reasoning done. Parenthesis add clarity and tell us what operations to do first.

**Two way street...**

We tell letter carrier stories and help our students write the numbers and the signs for the stories that we tell. And, as we did for word problems (multiple chapter and page 000 references), we also present the numbers and ask our students to create stories that match the numbers and the signs. Numbers for the words, words for the numbers—two way street.

**Negative numbers are...**

Negative numbers are as real as freezing temperatures on an icy day, and as much a part of daily life as an overdrawn checking account or our national debt. Negative numbers are a part of life. Negative numbers are a part of algebra.

## Lesson Four

| | |
|---|---|
| Purpose | Learn to write tables for pairs of numbers that are related in a patterned way. Learn to write equations or formulas for the patterns. |
| Summary | Students use a "Magic Box" function machine to predict from numbers going in, the numbers coming out. |
| Materials | "Magic Box" function machine, teacher-made and student-made "magic" cards. |
| Topic | Teacher creates the rules for the numbers going in and coming out. |
| Topic | Students create the rules for the numbers going in and coming out. |
| Topic | Write formulas for the rules. |

**Functions and variables...**

A function is a mathematical relationship paring two number sequences in which any term in the first sequence determines exactly one term in the second. The two paired numbers in the sequence are called variables. The first variable is called the independent variable because its value can be just about anything we choose. The second variable is called dependent because its value depends on the value we have chosen for the first. The relationship between the two variables is called a function.

A function may be represented by:

An equation or a formula—as in Lesson One.
A table—as in Lesson Four at hand.
A graph—as in Lesson Five and beyond.

The paired numbers of functions help us convert algebraic equations and formulas into visual pictures of all the possibilities. Since functions lead to pairs of numbers and pairs of numbers are something we can plot, functions link equations to Descartes' coordinate graphs.

To graph a function, it is first necessary to write a table of the numbers that are paired. The pairs of numbers become coordinates for the graph. To introduce our students to writing tables for functions, we use a function machine, or Magic Box.

**Function machines...**

In *Mathematics Their Way*, the Magic Box introduced on pages 248-249 is a function machine. A framework for introducing function machines is also contained in *Mathematics...a Way of Thinking*, Lessons 18-1 through 18-4.

In Lesson 18-1 the teacher creates rules for the function machine—a box with *in* and *out* slots. Stacks of pre-made cards inserted in the in-slot slide out the out-slot to introduce the students to independent and dependent pairs of variables. The number *in* is "changed" by the machine by some predictable rule or function. Students predict what the number *out* will be when they know the number *in.*

Once the students understand that the numbers in and out for each new stack of cards are connected by consistent sets of rules—one rule per stack of cards, they make up their own rules or functions from which their classmates may predict. In Lesson 18-2 the students make their own stacks of cards. The teacher then passes the student-made cards through the machine, as students predict a number coming out, for each number going in.

When use of the function machine is understood, the numbers going in and coming out are recorded in tables headed by boxes (for the numbers going in) and triangles (for the numbers coming out). Lessons 18-3 and 18-4 introduce the tables used for recording numbers in and out.

When we feel our students understand the use of boxes and triangles to represent numbers that go in and numbers that come out, we substitute the letters x and y for the boxes and triangles.

(illustration 14-4-1)
(Show the number machine box, and three different data tables—the box and triangle with in and out written in; the box and triangle with no in and out written in; and the data table with x and y replacing the box and triangle.)

**Writing formulas...**

Any function that passes through our machine can be recorded on a table headed by a box and a triangle, or by an x and y. If we know enough to predict the y number from the x, we know enough to write the rule.

(illustration 14-4-2)
(Tables from the illustration above with x and y in place of the box and triangle, and equations written beneath.)

If the rule is that the numbers coming out are twice as big as the numbers going in, then the formula or equation for the rule is: $y = 2x$. If the rule for the number coming out is that it is three more than the number going in, then the rule can be written: $y = x + 3$. If the rule is that the number out is twice the numbers in plus five, then the equation is: $y = 2x + 5$. Any table for a function can be represented by a formula. Algebra is how we write the rules for the patterns that we see.

**Lesson Five**

| | |
|---|---|
| Purpose | Learn to graph functions and the equations they represent. |
| Summary | Students plot the data from the tables in Lesson Four and other functional relationships on coordinate graphs and write equations to accompany their graphs. |
| Materials | Graph paper, circular objects, toothpicks. |
| Topic | $x = y = 10$. |
| Topic | Tables from Lesson Four. |
| Topic | Graphing the area formula, $a = lw$. |
| Topic | Graphing circumference, $c = \pi d$. |
| Topic | Graphing multiplication facts, $y = 2x$, $y = 3x$, etc. |
| Topic | Graphing toothpick patterns. |

**Ways to make ten...**

In *Beginning Addition and Subtraction* we explored ways to make eight with two groups of squares (page133). The comment was made then that the patterns we see in ways to make eight would be connected to equations in algebra that our students could plot on coordinate graphs. We'll use ways to make ten to see the connection to be made.

**Teacher: Give me two numbers that add to ten.**
**Student: Five and five.**
**Teacher: Okay. Give me two more.**
**Student: Seven and three.**
**Teacher: Okay. Let's plot the two sets of numbers that you gave me on a coordinate graph. Now let's connect the points and extend the line in both directions.**

(illustration 14-5-1)
(5, 5 and 7, 3 recorded on a data table headed by x and y. Coordinate graph with the two sets of numbers plotted with the line drawn in. The line extends into the negative quadrants.)

**Teacher: Give me another two numbers that add to ten.**
**Student: One and nine.**
**Teacher: Is the point one, nine on this line?**
**Student: Yes.**
**Teacher: Give me a way that uses fractions.**
**Student: Five and a half and four and a half.**
**Teacher: Is that point on this line?**
**Student: Yes.**
**Teacher: Can you think of any two numbers that add to ten that are not on this line?**

Our students might propose numbers like 100 and $^{-}90$. Would these points be on the line if we extended the graph far enough? Yes they would.