CHAPTER 2
Patterns on Number Tables

Lesson 2-1 page 5 0-99 matrix Students provide numbers to fill in blank matrix.	Lesson 2-2 page 7 0-99 matrix Blank matrix Teacher indicates spaces; students provide numbers.	Lesson 2-3 page 8 0-99 matrix Students search for patterns on matrix.	Lesson 2-4 page 10 Arrow Math 1-144 matrix One arrow rules discovered.
Lesson 2-5 page 10 Arrow Math 1-144 matrix More than one arrow rules discovered.	Lesson 2-6 page 11 Arrow Math 1-144 matrix Arrows off the matrix.	Lesson 2-7 page 12 Arrow Math 1-144 matrix ab$\boxed{}$cd shortest path.	Lesson 2-8 page 13 Arrow Math 1-144 matrix ab$\boxed{}$cd answer teacher questions.
Lesson 2-9 page 12 Arrow Math 1-144 matrix ab$\boxed{}$cd problems for others to solve.	Lesson 2-10 page 13 Arrow Math 1-144 matrix ab$\boxed{}$cd 5 arrow solutions	Lesson 2-11 page 14 Arrow Math Any matrix What difference does new matrix make?	

Prerequisite chapters:

None

MATERIALS

For overhead projector:

Transparencies ten by ten blank matrix		Worksheet 1
0–99 matrix		Worksheet 2
1–144 matrix		Worksheet 3

If no overhead projector is available:

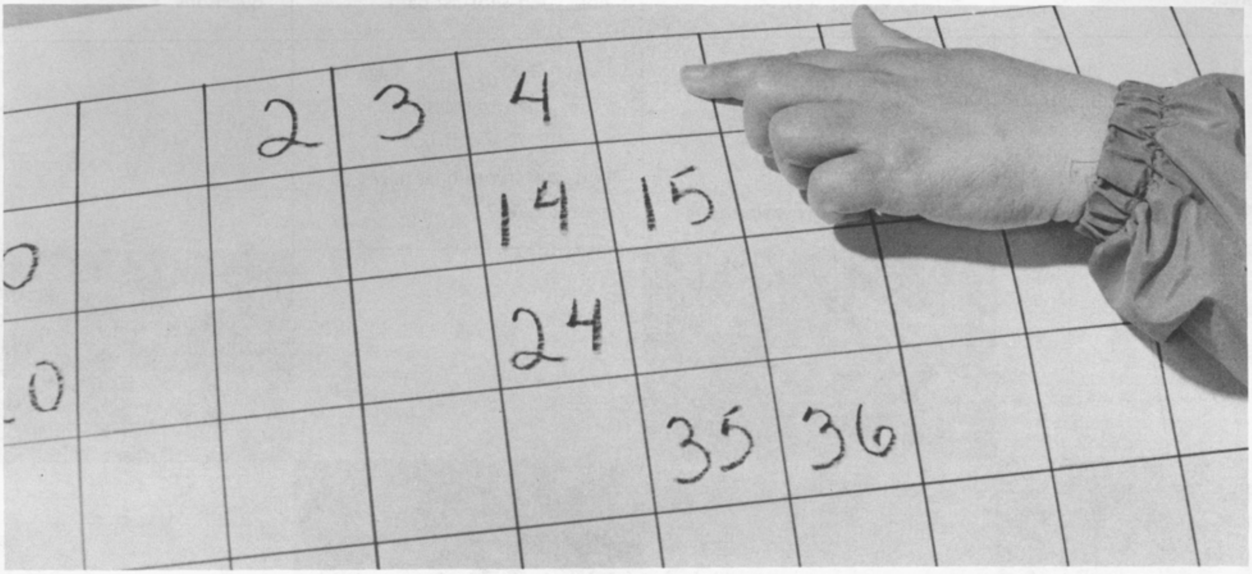
Make charts in place of the transparencies — Materials chapter, page 294

Student materials:

Dittos 1–99 matrix
1–144 matrix

Individual blackboards — Materials chapter, page 294

Unlined paper



The lessons in this book are based on the premise that patterns form the core of mathematics. The search for and discovery of these patterns are the foundations of the mathematical learning experiences described. Each mathematical exploration is undertaken with the assumption that within lies a pattern waiting to be discovered. The search may not always be successful, but it is always to be conducted.

The activities of this chapter familiarize the students with the process of examining tables of numbers in the search for patterns.

LESSON 2-1

0–99 MATRIX

PURPOSE:

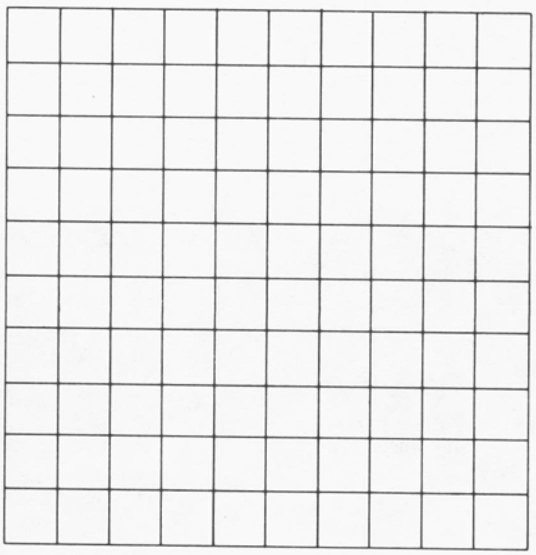
To provide experience in using number patterns to answer questions

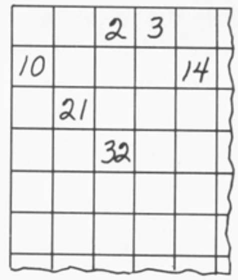
MATERIALS:

1. *Blank matrix ten squares by ten squares on a transparency, or blank matrix on a large tagboard*

In the following three lessons, students search for number patterns on an 0–99 matrix. The ability to see patterns provides students with a means of transferring knowledge from one situation to similar situations without having to relearn each new occurrence as if it were an unrelated, isolated piece of knowledge.

Teacher: I will write some numbers on this blank matrix.

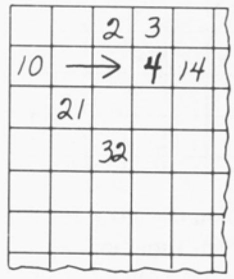




Now, can you tell me some other numbers to write in the other squares? The numbers I have written already are part of a pattern. If you see my pattern, you will be able to tell me what numbers are missing.

Student: Put a four by the three.

Teacher: Where? Here?

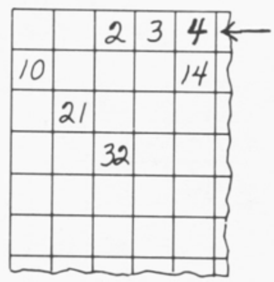


Student: No! By the three!

Teacher: Isn't this by the three?

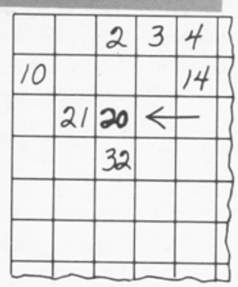
Student: No! Next to the three, above the 14!

Teacher: Oh!



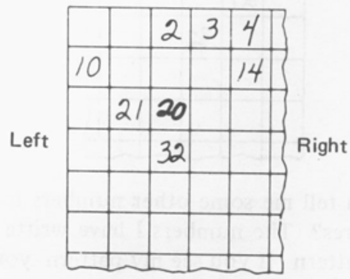
Learning need not be confined to the specific goals of the mathematics lesson at hand. A teacher should capitalize on any learning opportunity that arises. One of our goals is to help students refine their language and improve their ability to say what they want to say. If their instructions are unclear and we follow those instructions literally, our students can discover that what they meant may not have been what they said.

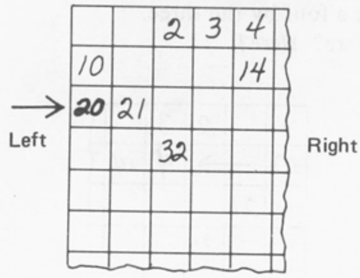
Student: Put a 20 next to the 21.



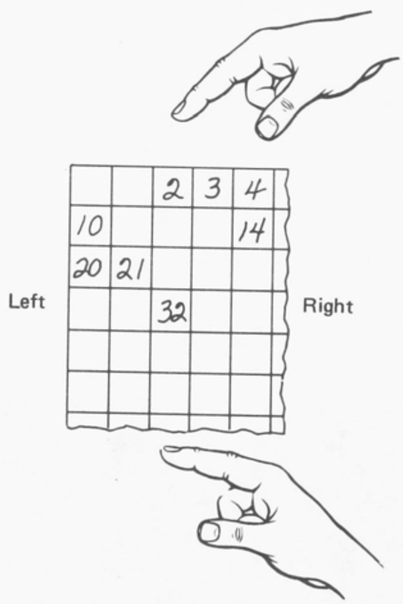
No! On the other side!

Teacher: It would help if you told me to the right or the left. I'll label this side 'right' and this side 'left,' so you can tell me which side you mean.

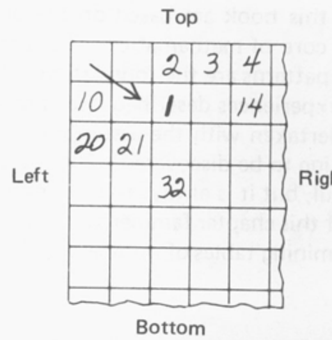




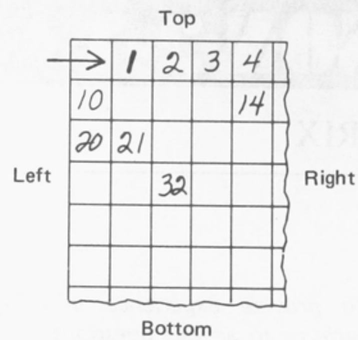
Student: Put the 20 on the left side of the 21.
Teacher: Could we put direction words like left and right here and here so you could tell me which way to put the numbers, this way too?



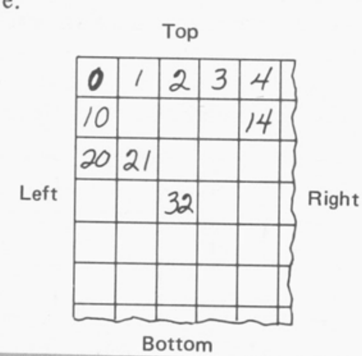
Student: Up and down.
Student: Top and bottom.
Student: Above and below.
Teacher: Okay class. Decide which you want me to use. How many want up and down? . . . Top and bottom? . . . Above and below? . . . Top and bottom wins.
Student: On top of the ten put a one.
Teacher: In some patterns the one could go on top of the ten, but in the pattern I'm using, I can't put a one there.
Student: Put the one next to the two then.



Student: I mean put the one on the left of the two!



Teacher: What number could I put in this first square in my pattern?
Student: Another one?
Teacher: In some patterns a one could go in there, but in this pattern I can't put another one in there.
Student: A ten?
Teacher: There could be a pattern that would put a ten in there, but I can't do that with the number pattern I'm using.
Student: A zero?
Teacher: Yes. In the pattern I am using, I can put a zero in there.



Whenever we get a response that isn't "right," we should avoid stating that it is "wrong." The answer given may not fit our number pattern, but the student may be thinking of an equally valid pattern of which we are unaware. Even if we believe the response is a wild guess from a bewildered person, it is still to our advantage to avoid saying the answer is wrong. The students we want most to involve in thinking and participating are those who hesitate to contribute their thoughts if it means risking failure. Wrong answers are failures.

The pattern of guessing continues until the entire matrix is filled in. At this stage, it is not necessary that every child understand the patterns used to place numbers in the squares. As a class, however, the students should experience little difficulty in filling in the table.

0	1	2	3	4	5	6	7	8	9
10	11	12	13	14	15	16	17	18	19
20	21	22	23	24	25	26	27	28	29
30	31	32	33	34	35	36	37	38	39
40	41	42	43	44	45	46	47	48	49
50	51	52	53	54	55	56	57	58	59
60	61	62	63	64	65	66	67	68	69
70	71	72	73	74	75	76	77	78	79
80	81	82	83	84	85	86	87	88	89
90	91	92	93	94	95	96	97	98	99

LESSON 2-2

0–99 MATRIX

PURPOSE:

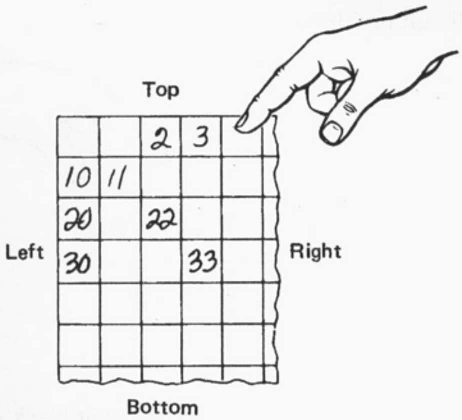
To provide a second opportunity to fill in the 0-99 matrix using number patterns

MATERIALS:

1. *Blank matrix ten squares by ten squares on a transparency, or blank matrix on a large tagboard*
2. *Individual blackboards*

Teacher: Yesterday, you filled in all the empty spaces on my table. You may not know it, but you were using patterns to help you fill in the spaces. You were looking at numbers you could see and using them to help you figure out what numbers were missing.

This time I'll write down some numbers on a new blank matrix. You show me by writing the numbers on your blackboards what you think goes in the spaces I point to. What number would go in this space?

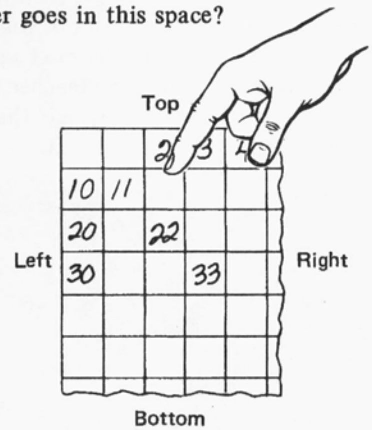


Student: Four.

Teacher: Yes, by *my* rule, four goes here.

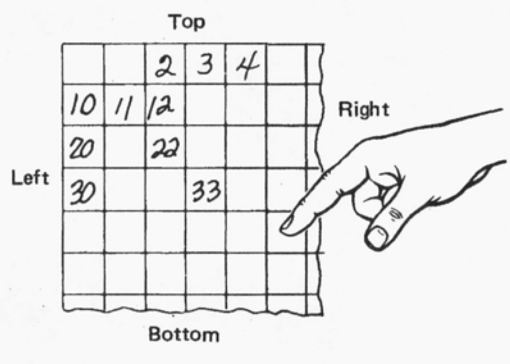
If you don't have four on your blackboard, then you are thinking of a different pattern. See if you can discover the pattern I am thinking of that would make me put four there.

What number goes in this space?

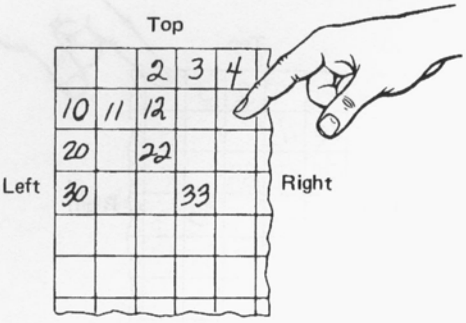


As the students respond on their blackboards, the teacher has the opportunity to check their level of understanding. If everyone holds up the right answer, a more difficult problem may be presented. If no one does, then the problem was too difficult or the instructions were unclear. The blackboards are the assessment tools for determining how difficult to make each new problem. More difficult problems are those in which the empty space to be filled in is not near existing numbers. Less difficult problems have an empty space next to or near a known number.

More difficult problem:



Less difficult problem:



The students write their guesses on their blackboards. At the teacher's signal, the blackboards are held up and the teacher writes the appropriate number in the space.

At no time should the teacher tell the class what the pattern is. Students must learn to rely on their own thinking to discover answers. However, the teacher can guide that discovery. As an example, the teacher places the numeral three in the appropriate square. Four is then written to the right of it. Next, the square to the right of the four is filled in. When the five has been added, the next square pointed to is to the right of the five. While the teacher refrains from explaining the pattern, the way in which the squares are filled enables more students to discover it.

It is often difficult to resist telling those few students who have not seen the pattern or guessed the rule what it was. The temptation to explain can be eased if we inform our students at the outset that we do not expect everyone to guess the patterns. They may be considering more patterns than we are and may not have come to ours yet. What we want them to do is think. Students who have not yet figured out the pattern have to think harder than those who already know it. They should be reassured that they are probably learning *more* from the lesson than anyone else.

The process of filling in the matrix is continued until all the squares contain numbers. The matrix produced from this lesson is the same as that produced in Lesson 2-1.

LESSON 2-3

0–99 MATRIX

PURPOSE:

To provide the opportunity to describe as many number patterns as can be found within the 0-99 matrix

MATERIALS:

1. *0-99 matrix on a transparency, or on a large tagboard*
2. *Dittoed copies of 0-99 matrix*

Note: Some students have difficulty examining number tables for patterns if the tables are at a distance from them, so all students should have a copy of the 0-99 matrix for reference.

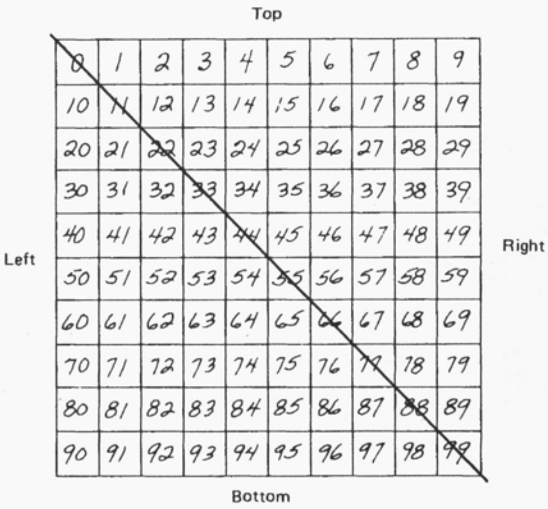
Teacher: You have already looked for number patterns on the zero to ninty-nine matrix. Today, I want you to tell me some of the patterns you have seen.

Students may be unable to respond to this statement, because they don't yet understand it. If so, their search may be guided in the following way:

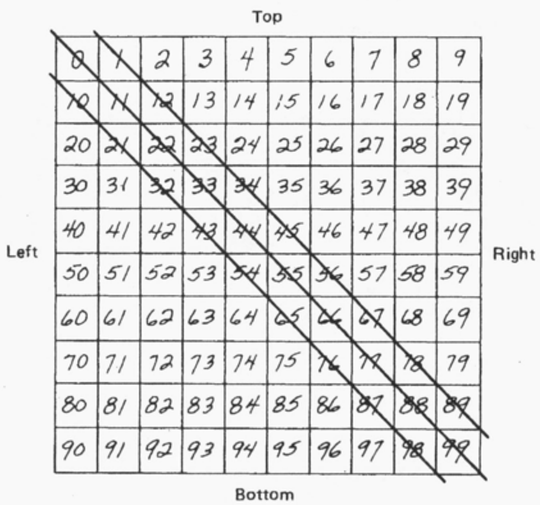
Teacher: What happens to the numbers as they go from left to right in the top row?
Student: They get bigger each time.
Teacher: Bigger than what?
Student: Each number is one bigger than the number next to it.
Teacher: Okay. Then the number pattern for the top row is that they get one larger each time. Is that same pattern true for the second row across?
Student: Yes.
Teacher: Any other rows across?
Student: The third . . . the fourth . . . the fifth . . . all of them.
Teacher: What happens to the numbers if we go from right to left across the top row?
Student: They get smaller by one each time.
Teacher: Which row across has numbers that don't get smaller by one each time?
Student: They all get smaller by one each time.
Teacher: Look at the numbers from top to bottom. Can you see any number patterns?
Student: In the first row there's a zero in every number.
Teacher: Where is the zero?
Student: On that side.
Teacher: Which side?
Student: The right side.
Teacher: Okay, in the first row down, all the numbers on the right have a zero.
Student: In the second row all the numbers on the right have a one.
Student: In the third row they're two, in the fourth they're three, in the fifth they're four . . . it goes up by one each time. All the numbers on the right going down are the same for each row.
Teacher: Is that only true going down? How about across?
Student: If you don't count the first row across, all the numbers on the left in a row are the same going across.
Teacher: What do you mean when you say they're all the same?
Student: They're all the same!

Teacher: Are they all twos?
Student: No! They're all the same in each row but each different row has a different number the same.
Student: And each row gets bigger by one.
Teacher: Where?
Student: On the left side.
Teacher: What else can you see?
Student: If you go from the top corner to the other bottom corner, all the numbers are doubles.
Teacher: Which top corner?
Student: The top left corner and the bottom right corner.
Teacher: What do you mean by doubles?
Student: One, one; two, two; three, three; . . .

If it helps to clarify what one student is saying for the benefit of the others, the teacher may draw a line on the transparency or giant pocket chart. For example, to show the diagonal of "doubles" referred to, a line would be drawn on the 0-99 matrix as indicated in this figure.



Teacher: The line I have drawn is called a *diagonal.* A diagonal line is a line drawn from one corner to another when the corners are not next to each other. What about the lines next to the diagonal? Are they double numbers too?

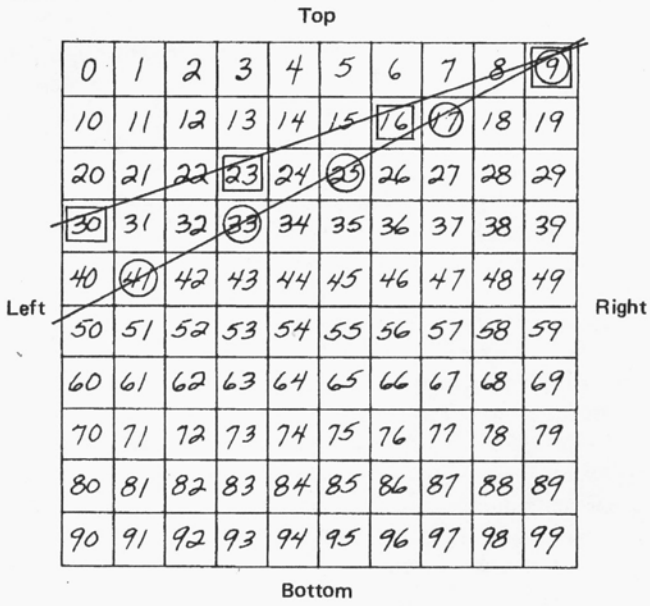


Student: No.
Teacher: What patterns do they have?

The questions the teacher may ask and the patterns the students may see are limitless. As the year progresses, the students will find number patterns within the matrix the teacher never imagined. It isn't necessary for the teacher to have seen all possible patterns before giving the students a chance to look. The teacher does, however, need to know what questions to ask to help the students see more patterns. Examples of questions are:

What happens to the numbers as we look in different directions along the rows, columns, and diagonals? How much larger or smaller do they get?
What happens to the pattern for the numbers if we look in the opposite direction?
Is there a row, column, or diagonal that gets larger or smaller faster than the row, column, or line next to it?
What happens if we look at the numbers in lines that are not straight up and down, or across, and that are not parallel to the diagonals?

Two examples of this kind of "line" can be seen in this figure.



No preconceived notions should be held as to what students must see. They are not supposed to *see* anything in particular, only to *look.* Students cannot be expected to recognize a pattern everytime they look. But, they will see many more patterns if they look for them than if they do not.

LESSON 2-4

ARROW MATH

PURPOSE:

To anticipate answers to simple Arrow Math problems by discovering the unannounced rules used by the teacher

PARALLEL PURPOSE:

To have the reader learn the rules of Arrow Math without an explanation

MATERIALS:

1. *1-144 matrix on a transparency or on a large tagboard*
2. *Dittoed copies of 1-144 matrix*
3. *Individual blackboards*

In the next three lessons students learn the rules for the game called Arrow Math without ever being told what they are. To allow the reader to observe first hand that such learning is possible, the reader is taught in the same manner as the students would be.

You will not be given any more information than you should give your students, nor will you be told at the end of this presentation if you have figured out the game correctly. Both you and your students can decide if you really understand.

Refer to the 1-144 matrix in the figure above to answer the questions you are asked. Write your answers, then compare with the answers the teacher gets.

Teacher: What number do you think I mean by "20→"? Look at the matrix and write your guess down.

20→ could mean many different things. You may be thinking of a more interesting rule than I am. But, the rule for this game is that you must figure out what *I* mean.

According to my rule, 20→ means 21.

Whatever you thought 20→ meant, and it could be many things, you now know that my rule gives a 21. I will not tell you my rule, but here is another problem to consider.

What do you think I mean by 76→?

You might think easier if you cover this page below the line being read so you can guess what my rule is before you see my answer.

By my rule 76→ means 77.

If you didn't predict 77, then find 76 on your matrix and decide what rule I might have used to get 77.

Try this: 47→.

You already have come to some conclusions about what my rule might be. You may not be surprised to find

133	134	135	136	137	138	139	140	141	142	143	144
121	122	123	124	125	126	127	128	129	130	131	132
109	110	111	112	113	114	115	116	117	118	119	120
97	98	99	100	101	102	103	104	105	106	107	108
85	86	87	88	89	90	91	92	93	94	95	96
73	74	75	76	77	78	79	80	81	82	83	84
61	62	63	64	65	66	67	68	69	70	71	72
49	50	51	52	53	54	55	56	57	58	59	60
37	38	39	40	41	42	43	44	45	46	47	48
25	26	27	28	29	30	31	32	33	34	35	36
13	14	15	16	17	18	19	20	21	22	23	24
1	2	3	4	5	6	7	8	9	10	11	12

the answer for this is 48. If you don't know where the 48 came from, look at your 1-144 matrix and think about it.

How about 52↑.

The arrows can change direction. Many possibilities exist for 52↑, but when I tell you the answer, you will be more able to figure out my rule.

For my rule, 52↑ means 64.

Try: 91↑.

If you don't understand 52↑ being 64, study that on your matrix first.

By my rule, 91↑ means 103.

Have you guessed my rule? Don't tell me if you have, because I will not ask you to state it. I will watch, instead, to see if you can predict my answers.

In the classroom, we would give our students the problems one at a time. We would also give them time to ponder our answer if it differs from what they thought. As in any problem situation, we stay at one level of difficulty until most, but not necessarily all, of our students can predict the answer that our pattern or rule gives for the problem.

LESSON 2-5

ARROW MATH

PURPOSE:

To anticipate answers to Arrow Math problems that contain more than one arrow

PARALLEL PURPOSE:

To have the reader continue to discover the rules of the Arrow Math game

MATERIALS:

1. *1–144 matrix on a transparency, or on a large tagboard*
2. *Dittoed copies of 1–144 matrix*
3. *Individual blackboards*

Teacher: Each problem in Lesson 2–4 used only one arrow. Now you will try problems that use more than one arrow at a time. Try the problems shown at left and cover the answers at right until you are ready to check your work. Then, compare your answers with the answers given.

57 →	58
76 ↑	88
18 ←	17
43 ⇉	45
69 ↓↑	69
90 →→←	91
101 ↓→	90

If you disagree with some of the answers given it doesn't mean you have not thought of a good rule, only that you have not figured out *my* rule. Look at my answer then study the matrix to see what I have decided the arrows mean.

Try this one: 27↗.

Now the arrow points in a different direction. Does your knowledge of the arrow from problems like 20→ help you discover a meaning for 27↗? Can you know for sure what 27↗ means before I tell you what my rule says it means?

By my rule 27↗ means 40. Look at your matrix and see why.

Now look at the new problems shown above right. Use your matrix to help you work each problem. When you think you have the answers my rule would give, look at those shown at the right and check your work.

When you have checked your answers against mine, use the matrix to see how I got my numbers, if you have any answers that disagree with mine because you were thinking of a different rule. If your answers agree, you have probably figured out all the rules I have been using for the arrows, even though I have told you none of them.

3 ↗↗	29
17 ↗↗↗	56
64 ↗↙⇄	64
90 ↗↘↙↖	90
118 ⇉↓	108
112 ⇉↓↓↘	79
127 ↙↙↙↙↓	63

Reemphasizing, in a classroom setting we present the problems to our students one at a time, allowing those who don't get the answer time to examine the 1–144 matrix to attempt to discover our rule.

LESSON 2-6

ARROW MATH

PURPOSE:

To anticipate answers to Arrow Math problems containing arrows that point off the matrix

PARALLEL PURPOSE:

To allow the reader to continue making discoveries

MATERIALS:

1. *1–144 matrix on a transparency, or on a large tagboard*
2. *Dittoed copies of 1–144 matrix*
3. *Individual blackboards*

Teacher: I have purposely allowed the path of the arrows to keep you on the matrix. Now, you are more advanced. You are ready for problems like 72→.

What do you think 72→ means? Good arguments could be made for answers 61, 73, 72, and many others. Look at the numbers 61, 73, and 72 and see if you can state different rules that would make each one a plausible answer to 72→.

If you have good reasons for your three answers, you can see that my answer is not the only one. Once I select an answer, however, even if you find another more plausible, my choice defines the rule of this game. If my answer were a number other than 61, 73, or 72, you would have to do more thinking to discover my rule.

In this case, by my rule, 72→ means 73.

Even though I have not told you what my rule is, the answer 73 directs your thinking in a very specific way. Would you now be surprised if I stated that 108→ means 109?

Try the problems shown at left. When you think you have successfully anticipated all the answers, check your predictions against the numbers shown at right.

96 →	97
84 ↗	97
47 ⇉	49
25 ←	24
24 ↗↙ ↓↑	24
109 ⇇ ↓↓	83

For a little more challenge:

140 ↗ ⇉	155
132 ↗↗ →	159

For a lot more challenge:

5 ↓	−7
8 ↓↘	−15
12 ↘↘↘	−21

Do not worry if you did not always figure out my answer in advance. The problems can go from a relatively easy 5↑ to a relatively hard 5↓. Maybe you hadn't discovered what I had each arrow stand for; that is, you used the arrows for direction indicators and I used them as numbers, or vice versa. When I use the arrows to carry you off the matrix, your understanding of what you *think* I have been doing gets a real test.

Are you sure you know my rules? Did you always guess right? If you didn't, were you able to use my answer to find a new reason for why I got that answer? Did you have to think? Did you have to look at the numbers to figure out what was going on? Did you really believe me when I said I wouldn't tell you my rules?

Initially, when Arrow Math is used with students, the teacher selects problems that keep the students on the matrix. The teacher may also control the level of difficulty by using the same arrow paths for several problems in a row, for example, 53↓→, 26↓→, 103↓→. Problems that go *off* the matrix are not introduced until the class as a whole has demonstrated competence *on* the matrix.

LESSON 2-7

ARROW MATH

PURPOSE:

To learn to construct Arrow Math problems

MATERIALS:

1. *1-144 matrix on a transparency, or on a large tagboard*
2. *Dittoed copies of 1-144 matrix*
3. *Individual blackboards*

In Lessons 2-4, 2-5, and 2-6 the teacher presented a problem, such as 55↗↗, and got an answer of 81. To teach the students to devise their own problems, the basic question must be rephrased.

Instead of writing 55↗↗, the teacher writes 55 ☐ 81, and asks:

How can you get from 55 to 81 using the fewest arrows? Draw the arrows that are your answer in the box between the two numbers.

The answer would be written: 55 $\boxed{↗↗}$ 81.

In the lesson, the teacher writes one problem at a time on the overhead projector or the blackboard. The students copy each new problem onto their blackboards. Using the 1-144 matrix, each student draws in the arrows representing the shortest path between the two numbers, then the teacher checks their answers.

Because the rules for Arrow Math are the teacher's rules, the teacher ends each problem by filling in the shortest path between the two numbers. When more than one "shortest" path exists, all are listed as possible answers.

LESSON 2-8

ARROW MATH

PURPOSE:

To investigate possible answers to questions relating to arrow paths

MATERIALS:

1. *Dittoed copies of 1–144 matrix*
2. *Unlined newsprint*

In the next three lessons students create and work their own problems. Students can realize that the creation of problems to be solved is not solely in the hands of the teacher. This notion will be expanded in many of the following chapters.

In addition, the students will continue to examine specific problems for patterns which might enable them to predict answers to problems not yet seen. This process will also be encountered in the chapters that follow.

Teacher: Yesterday, I had you put in the boxes the arrows that made the shortest path between the two numbers. Today, *you* will decide which numbers you want to use in some problems of your own. Then I will ask you to answer some questions about arrow paths.

The students pick sets of two numbers and write the shortest arrow paths between them. After they begin working, they are asked to think about answers to the following question:

Pick two numbers so that you could get from the first to the second using only two arrows.

Is there a path between your two numbers that could take three arrows? Four arrows? Five?

What is the longest arrow path between your two numbers?

Does the order in which you do what the arrows ask make any difference? If you have a path that is written ↗↗→, can you travel the path as if it were written →↗↗ or ↗→↗? Do all these paths lead to the same place?

Is there a number you can't get to from another number using arrows?

LESSON 2-9

ARROW MATH

PURPOSE:

To construct Arrow Math problems for others to solve; to solve Arrow Math problems constructed by others

MATERIALS:

1. *Dittoed copies of 1–144 matrix*
2. *Unlined newsprint*

Teacher: Take out your blackboards, please. Draw a box like the ones we used yesterday. Now, pick two numbers from your 1–144 matrix. Put one at the outside of one end of your box; put the other at the other end. Hold up your blackboards so I can see how clear my instructions were.

Okay, you have just made up an Arrow Math problem. Now, make up a page of numbers and boxes but don't put any answers in the boxes . . . as you did just now on your blackboards.

When the problems have been devised, the students exchange papers then figure what arrows fit into the empty boxes. The student who made up the problems is responsible for checking the accuracy of the answers. For this lesson, the requirement is that the arrow paths must be the shortest route between the two numbers on either end of the box.

LESSON 2-10

ARROW MATH

PURPOSE:

To continue constructing Arrow Math problems for others to solve; to investigate possible answers to questions relating to arrow paths

MATERIALS:

1. *Dittoed copies of 1–144 matrix*
2. *Unlined newsprint*

Teacher: Today, I want you to make up Arrow Math problems to exchange with one another the same way we did yesterday. After you trade papers, see if you can find arrow paths between each two numbers that use five arrows—no more and no less.

Once the students have begun to search for solutions to each other's problems, the teacher asks them to think about the answers to the following questions:

Do all the problems have solutions that can be expressed in terms of exactly five arrows? Why? Why not?

Is it possible to predict in advance if a problem will have a five-arrow solution?

How can we be sure that if we couldn't find a five-arrow solution, no one else could either?

LESSON 2-11

ARROW MATH

PURPOSE:

To expand the knowledge of Arrow Math gained on the 1-144 matrix to matrices containing other number arrangements

MATERIALS:

The teacher decides what materials are necessary

The Arrow Math activities described in Lessons 2-4 through 2-10 have made use of the 1-144 matrix. However, the specific matrix used is not important. The 0-99 matrix, addition or multiplication tables, monthly calendars, and matrices whose rows and columns of squares number other than ten would work equally well.

The teacher presents these questions:

Does changing the matrix give new meaning to the arrows?
What does 20→ mean on the 0-99 matrix? How about 27↗?
Are the numbers the arrows produce on a new matrix the same as they were on the 1-144 table?
If the numbers change, has what the arrows mean changed as well?

Do you, the reader, think you have learned enough about the arrows and how to use them on the 1-144 matrix to use the same arrows on a 0-99 matrix? How did you learn as much as you did when you were never told the rules?

This chapter has demonstrated that students can learn for themselves without being told the rules that govern the lessons. The following chapters further develop this theory.