CHAPTER 4
Beginning Addition and Subtraction

Prerequisite chapters:

None

MATERIALS

For overhead projector:

Tiles _____ Materials chapter, page 294
Cubes _____ Materials chapter, page 295

If no overhead projector is available:

Square shapes _____ Materials chapter, page 295

Student materials:

Tiles
Cubes
Individual blackboards _____ Materials chapter, page 294
Lined paper

The activities in this chapter provide students a framework for creating and checking their own addition and subtraction problems. Although the arithmetic problems created by the students may involve large numbers, the concept of place value is reserved for the chapter on advanced addition and subtraction.

LESSON 4-1

ADDITION WITH TILES

PURPOSE:

To create and record simple addition problems with tiles

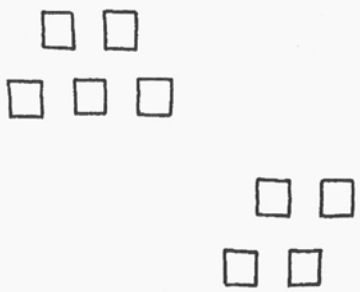
MATERIALS:

1. *If no overhead projector is available, square shapes*
2. *Tiles*
3. *Individual blackboards*
4. *Unlined paper*

This lesson presents students with a technique of creating addition problems, the answers to which they check themselves. This frees the teacher from having to provide students with pages of problems and having to check each student's completed work.

Teacher: I want you to take a handful of tiles and put them in a pile on your desk. Now, count the number of tiles in your pile.

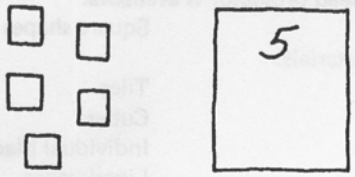
Take another handful and make a second pile, being careful not to mix it up with your first pile. Count how many tiles you have in your second pile.



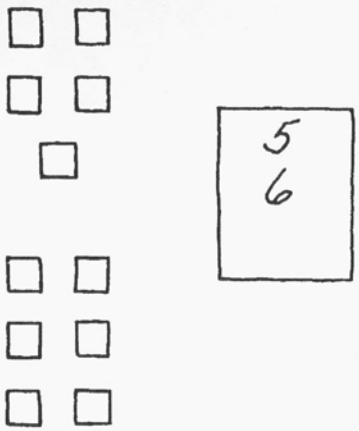
Push the two piles together and count how many tiles you have altogether.

This process is repeated until the students can take two handsful, one at a time, count them separately, push them together, and count the total. The teacher has no way of knowing, initially, if everyone is counting correctly, but the emphasis at this point is on the precedure, not on accuracy.

Teacher: This time take one handful of tiles, count it, then write the number on your blackboard. Hold up your blackboards so I can see how clear I made my instructions . . . Good!



Now, take a second handful and put it on your desk. Remember not to mix it with your first handful. Count the tiles and write the number on your blackboard underneath the first number. Hold up your board so I can see if my instructions are still clear.



Before we started writing numbers on the blackboards, what did we do after counting the second pile?

Student: Pushed them together and counted them.

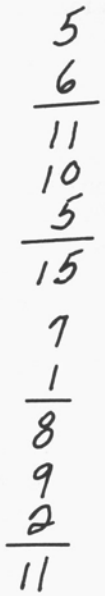
Teacher: Then do that now. Push both groups together and count the tiles you have altogether. Write that number underneath the first two numbers. So I can tell the last number you write down is not just another pile, I want you to draw a line under the number for the last pile you counted before you pushed them all together.

When you've drawn the line, write the total number of tiles you have beneath the line, like this.

The initial recording is done on individual blackboards so the teacher may tell at a glance how well the students understand the process. The blackboards offer the advantage of pinpointing those students who need assistance. The teacher can also tell if a general pattern of confusion exists or if only a few students need special help.

Once students are successfully recording problems on their blackboards, they can transfer to paper. The teacher explains that recording the problem on paper provides a way for the teacher to follow the progress of each student, and lets the students share with their teacher what they have accomplished.

Writing numbers on paper is a very different experience from writing problems on a blackboard. With the blackboard, each problem is erased before a new one is written, so spacing is not an important consideration. On paper, however, students are apt to record their numbers as shown.



Teacher: I have trouble telling what problems people have done when they record their work on paper. All the numbers run together in a long line and I can't tell which problem is which, or where one problem stops and another begins. Is there something we can do about this?

Student: Put a circle around the answer.

Student: Put a space before each new problem.

Student: Put a box around the whole problem.

Student: Make a rule that the answer always comes under the line, so you'll know where it is. Then you'll know where the next problem starts.

Teacher: Each of your suggestions might allow me to find your answers. I want each of you to decide which way you will use—one of the suggested ways or a way of your own. Whatever you pick will be okay with me, as long as I can tell where your answer is.

We were taught at different times in school that the *only* acceptable way to designate problems was to double underline, circle, or box the answer, do nothing with the answer—just leave a big space, or write Q.E.D. by the answer.

And there are many more ways, each of which may have been taught as the only way.

Our teachers taught us the "only" way to make sure our problems were in a readable form, so the answers could be found quickly. We, too, want to find the answers quickly, but we have another goal as well. We want to give our students as much opportunity to develop their thinking skills as possible.

The ability to think is the most valuable skill we can pass on to our students, far more important than any particulars of knowledge. If they learn nothing more than how to think for themselves, they have learned a great deal.

If our goal is to let students think for themselves, we must constantly change how we pose problems. We must switch from situations where we see the problem and dictate the solution, to expressing the problem and soliciting from our students solutions that deal with it effectively.

To this point, the students have learned how to use handsful of tiles to find numbers to write on their papers, thus creating their own addition problems. The teacher's responsibility at this time is to circulate among the students, checking to see each one understands the process of creating and recording addition problems. Each student is free to create problems of any size he or she wishes.

Each problem created, regardless of its size, is accepted equally by the teacher. Within the framework of any lesson, *all* students should have the opportunity to feel successful.

The written numbers are only a record of what each student has done with the tiles. At this point nothing guarantees that every student can count accurately enough to produce correct answers to the problems created. It may be assumed, however, that if a student counts and records correctly, all the answers are correct. Consequently, the teacher does not have to check the students' work.

The children we teach come from an assortment of school backgrounds. They have one thing in common, though—they expect us to tell them the answers, to pass final judgement on their work. It is more beneficial if we help students learn to rely on their own talents rather than ours.

We can provide students with the power to know, and to know when they know, and we must help them make full use of this power. Whatever abilities we are able to instill in them will remain with them—we won't.

When our students ask if an answer is right, we must ask them in turn:

Did you work the problem carefully?

If you think you've made a mistake, then do the problem again to check your work.

If you think your work is right, then *you* can tell *me* if your answer is right.

The more we allow our students to rely on their own intelligence in finding answers to problems, the less difficult teaching mathematics or any other subject becomes. By allowing them to accept responsibility for the correctness of their work, we help create in our classrooms an environment that encourages thinking.

LESSON 4-2

ADDITION WITH TILES

PURPOSE:

To explore answers to teacher-directed questions while creating and recording simple addition problems with tiles

MATERIALS:

1. *Tiles*
2. *Unlined paper*

Once students understand the basic procedures for creating and checking their own addition problems the teacher expands the range of problems by asking questions. This permits each student to operate at his or her level of ability while all students are using the same materials.

The students continue creating and recording their own addition problems with tiles. Because of the students' responsibility for their own work, the teacher is free to observe them individually.

This allows the teacher to become aquainted with each student and observe how each approaches problem solving situations.

As students work on the basic task of constructing addition problems, the teacher may ask individual students or the whole class to begin work on a greater assortment of tasks. Everyone need not work on the same kind of problem once the general framework for constructing and recording them has been learned.

Teacher: If you think you have created enough two-pile problems, you may try making up problems that use three handsful. Or, if you prefer, continue what you are already doing.

Advancing to a more difficult level of problems should be an individual choice based on what each student feels. The suggestion to try something else should be phrased in a way that leaves students with a real choice. Moving from one level to another should not become a status symbol worthy of special praise. Some students will need to stay with two piles for a long time, others will choose to devise problems involving multiple handsful. Contentment at one's own level of success can only exist if no pressure exists to advance quickly to the next.

The suggestions the teacher makes to individuals or to the whole class about a possible new direction are phrased in a soft-sell manner. It is not important if students change what they do on any given day—learning will continue to take place. Either level of difficulty will provide them with sufficient skill to become successfully involved in the lessons that lie ahead.

The teacher might pose the following kinds of exploratory questions:

Can you make up problems for three handsful?
If you've done three, how about four? . . . Five? . . . Six? . . .
What is the largest number of groups you could make up problems for?
What is the largest problem you could make with your tiles?
Before you push the tiles together and count them, predict how many you think you will have altogether. Write down your prediction. Count your tiles and check your prediction.

Any question that seems worth pursuing should be asked. Ideas for questions may come from observations the teacher makes while assisting individual students, or from suggestions students make about things they'd like to try. Whenever the teacher notices a student whose curiosity carried him or her in a new direction, that direction is mentioned to the whole class. Anyone who wishes may go off on the same tangent. The goal is to provide a classroom environment where students who need the security of simplicity and those who find joy in challenge are treated as equals.

No particular number of problems must be completed within the time period for the lesson, so students may choose to construct many small problems, a few large problems, or something in between. All are creating their own addition problems, and all can feel successful.

LESSON 4-3

SUBTRACTION WITH TILES

PURPOSE:

To create and record simple subtraction problems with tiles

MATERIALS:

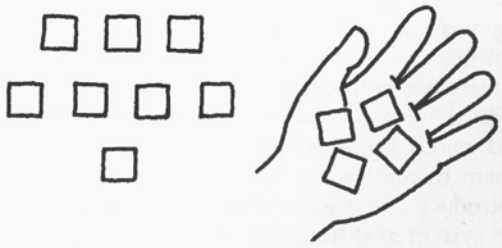
1. *If no overhead projector is available, square shapes*
2. *Tiles*
3. *Individual blackboards*
4. *Unlined paper*

The next two lessons present students with a technique for creating subtraction problems, as in the previous two lessons on addition.

Teacher: Take a big handful of tiles, put them on your desk, and count them. This time I don't want you to take another handful. I want you to take in your hand some of the tiles you already have in your pile, then count them.

When you have finished counting the ones in your hand, put them back in your tile box.

Now count how many you have left on your desk.

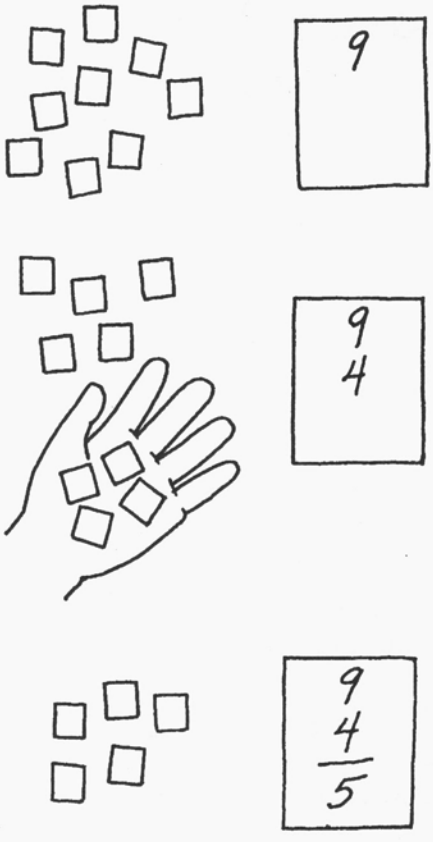


This process is repeated until these three separate counting instructions are clear to the students: (1) count the whole pile, (2) count the ones taken away and held in the hand, and (3) count the ones remaining.

Teacher: Take a big handful, count how many you have, and write the result at the top of your blackboard. Now, take some away from the pile on your desk, put them in your hand, and count them.

When you've finished counting the tiles in your hand, write down the number. Put those tiles back in your box and draw a line underneath the two numbers you have so far.

How many do you have left? Write that number underneath the line.



The steps are recorded on the individual blackboards until the students demonstrate an understanding of the process. If a substantial majority of the students understand, the whole class may begin recording on paper. The teacher then moves around the room assisting those students for whom the instructions were less clear. If about half the students understand the process, the teacher asks them to assist in making the instructions clear to everyone by pairing up with those who don't understand. The teacher then explains the lesson again while the teams assist each other in making the teacher's meaning clear. If very few students understand, the teacher begins again, using different descriptive words and going at a much slower pace.

LESSON 4-4

SUBTRACTION WITH TILES

PURPOSE:

To explore answers to the teacher's questions while creating and recording simple subtraction problems with tiles

MATERIALS:

1. *Tiles*
2. *Unlined paper*

As the students create and record subtraction problems with tiles, the teacher presents questions to individuals or the whole class that may lead them to explore a greater range of problems. The students are free to explore the answers, create their own questions, or continue making up problems as in Lesson 4–3.

Examples of possible questions are:

What's the largest problem you can make up?

What's the biggest pile you can start with and the smallest pile you could take away?

What's the biggest pile you can start with and the biggest amount you can take away, and still have something left?

What happens when you take away zero? Does the same thing happen for other piles? Find the pile where the same thing *doesn't* happen.

Put out your starting pile, then take some away. Predict how many you think you have left on your desk *before* you count them. Write down your prediction. Count the remaining pile and check your prediction.

LESSON 4-5

ADDITION AND SUBTRACTION WITH TILES

PURPOSE:

To create and record simple addition and subtraction problems with tiles

MATERIALS:

1. *Tiles*
2. *Unlined paper*

The students have already made several pages of either addition or subtraction problems. In this lesson, they construct problems of each type alternately. The teacher demonstrates two or three examples of alternating problems, then the students begin working on their own.

Shortly after the students begin work, the teacher stops them and explains a difficulty that has arisen.

Teacher: How can I tell which problems resulted from your pushing piles of tiles together, and which resulted from your taking tiles away?

Student: The answers for pushing the tiles together are bigger than for taking away.

Teacher: How about this?

$$\begin{array}{r} 3 \\ 4 \\ \hline 7 \end{array} \qquad \begin{array}{r} 15 \\ 2 \\ \hline 13 \end{array}$$

Student: For adding together, the answer is bigger than the top number. For taking away the answer is smaller.

Teacher: What are these then?

$$\begin{array}{r} 15 \\ 0 \\ \hline 15 \end{array} \qquad \begin{array}{r} 15 \\ 0 \\ \hline 15 \end{array}$$

Student: Except for zero, those could be both ways.

Teacher: How can I tell which way you meant? How about these?

$$\begin{array}{r} 15 \\ 11 \\ \hline 30 \end{array} \qquad \begin{array}{r} 12 \\ 4 \\ \hline 4 \end{array}$$

Student: Those are wrong.

Teacher: You may think they're wrong, but later you'll be learning to do arithmetic in different bases and you may get answers like these with your tiles you think are right.

How can I tell from your paper whether you pushed tiles together or took some away?

The students might suggest almost immediately that plus and minus symbols be used. However, they might spend considerable time inventing their own systems for distinguishing one kind of problem from another.

The teacher's purpose in presenting the "difficulty" to the students is to provide a justification for the use of plus and minus symbols. If the students don't suggest using them to resolve the confusion, the teacher may always introduce the two symbols as the most commonly accepted system of differentiation.

The teacher may also allow the students to invent any designating system they wish, the requirement being that it clearly distinguish the two types of problems. This system would eventually have to be adjusted to clearly accomodate multiplication and division.

Once the dilemma of how to tell one kind of problem from another has been presented and dealt with, the students proceed to construct problems of both kinds, within the time limit of the lesson.

LESSON 4-6

ADDITION WITH TILES

PURPOSE:

To provide practice in addition, while searching for patterns in numbers

MATERIALS:

1. *If no overhead projector is available, square shapes*
2. *Tiles*
3. *Individual blackboards*
4. *Lined paper*

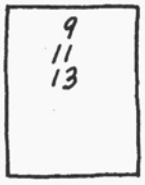
In the next four lessons students combine their work done with patterns in Chapters 2 and 3 with their ability to construct addition and subtraction problems to explore patterns produced by repetitive addition or subtraction of constant numbers. These pattern explorations provide motivation for drill in simple addition and subtraction. Many of the patterns seen will be in the students' later work with multiplication, division, and fractions.

Teacher: I am putting a handful of tiles on the overhead. How many do you see?

Student: Seven.

Teacher: Okay, put seven tiles on your desks. I want you to add two more tiles to what you have on your desks and write down the new total on your blackboards.

Add two more. Count how many you have now, and write it down. Add two more . . . count the total . . . write it down.



The teacher continues the sequence through four or five more additions of two. At each step, the students keep track of the new totals on their blackboards. When two has been added several times, the teacher asks the students to predict the total before the next two tiles are added. The two tiles are then added to check the accuracy of the prediction. When the list is about ten numbers long, the teacher asks the class questions like the following:

Look at the numbers we have recorded . . . can you see any patterns in the numbers in the right-hand column?

If we add more tiles, would the patterns continue?

How long do you think the pattern will continue? Will it change?

Could we use the patterns we see to make predictions about the numbers we will get as we keep adding two tiles at a time?

Students do not need to work out many examples to verify their predictions. They should, however, check their prediction with tiles until they are convinced they have enough evidence to support its accuracy.

The students and the teacher work several examples together; in each, the initial pile is different and two tiles are repeatedly added to it. When the teacher feels the students understand the process, they create problems independently.

The students record their work on lined paper, starting at the top and adding twos until they have numbers on every line, top to bottom. This is to encourage them to look for patterns. Students who dutifully add two each time, counting to find the new total, can add repeatedly until they reach the bottom. The students who examine the numbers being recorded for patterns may get to the bottom with considerably less effort.

When students reach the bottom of their first "add two" column, they may choose between starting with a new handful and adding twos, or adding threes for a whole column. This continues with fours, fives, and so on, as time permits. Irrespective of the constant number added, the search for patterns is the same.

LESSON 4-7

ADDITION WITH TILES

PURPOSE:

To explore answers to teacher-directed questions while searching for patterns in numbers

MATERIALS:

1. *Tiles*
2. *Lined paper*

The students continue the process of adding a constant number of tiles to an initial handful. As they work, the teacher presents questions to be considered, as in Lesson 4-2. Examples of possible questions are:

How many numbers are in the right-hand column before the pattern repeats itself? Is this number the same for all patterns, or does it change depending on whether we add twos, threes, or fours?

Does it make any difference in a pattern what number you had in the handful with which you started?

Can you tell by how much you are adding each time, what kind of pattern to look for? Can you tell by the number in your first handful?

Within the framework of the lesson, students will operate at a wide variety of levels. To match this diversity, the teacher may ask a question designed to challenge a quicker student, and one to a slower student focusing on the patterns within the numbers being recorded.

After students have generated number columns for twos, threes, fours, and so on, the teacher suggests that they explore patterns for adding one each time. They have seen the counting numbers many times, but searching these same numbers for patterns is a fascinating experience. Most students have never noticed the predictable regularity in numbers as they advance by ones. Questions such as What comes after 39? or, Is 100 after 89 or 99? come from students who have never had the opportunity to see that what comes before and after a number is a part of a beautifully simple pattern. When students learn to see patterns, they also learn they can count much better and much higher than they thought they could.

LESSON 4-8

ADDITION WITH TILES

PURPOSE:

To explore answers to teacher directed-questions, while searching for patterns in numbers

MATERIALS:

1. *Tiles*
2. *Lined paper*

Instead of beginning with a handful of tiles, the students start each column in this lesson with zero.

When all columns start with zero, the first column of numbers resulting from the repeated addition of two tiles produces the number pattern shown.

0
2
4
6
8
10
12
14
16
18
20
22
24
26
28
30
32
34

Once the students have completed a column for adding twos, they may begin with zero and add threes. If any wish to start again with zero and add twos they may do so. When they see they are doing nothing new, they may then begin adding threes.

As the students examine their new patterns, the teacher again circulates among them, asking the same questions as in the previous lesson, leaving out the discussion of the effect of different starting amounts on the eventual patterns.

LESSON 4-9

SUBTRACTION WITH TILES

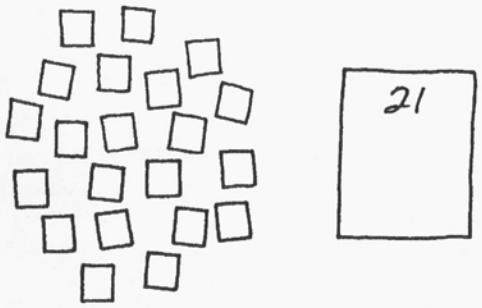
PURPOSE:

To provide practice in subtraction while searching for patterns in numbers

MATERIALS:

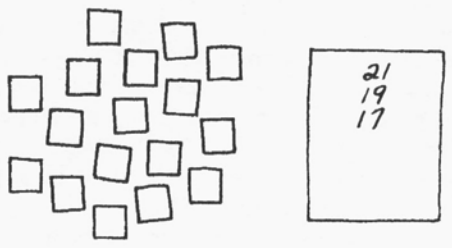
1. *Tiles*
2. *Lined paper*
3. *Individual blackboards*

Teacher: Take a lot of tiles, count them, and write down how many you have.



To develop a good pattern in subtraction, it is helpful to start with a large number of tiles. There are, however, students who cannot yet count well enough to set out large numbers of tiles accurately. For the beginning lesson in subtraction, therefore, the students pick their own starting piles.

Teacher: Now, take two away and return them to your tile box. How many do you have left? Count them and write down the number you have left. Take two more away. Put them in your tile box. Count how many you have left.



The students respond to the teacher's questions by holding up their blackboards. Since the students started

with different sized piles, a variety of numbers will be recorded.

As soon as the students understand the process of repeated subtraction, they begin recording their numbers on lined newsprint. As in the similar activities in addition, the teacher asks the students to examine the columns of numbers for patterns.

While subtracting constant numbers, the students may explore the answers to the following questions:

What happened to the numbers in the right-hand column when you were adding by twos each time?
Does the same thing happen when you subtract by twos?
How about adding and subtracting by threes? By fours?
What numbers do the same kinds of things whether you add or subtract?
What numbers do different things?
Do the patterns change if you start with a larger number of tiles?

LESSON 4-10

NUMBER FACTS WITH TILES

PURPOSE:

To provide experience in developing problems to match a given answer

MATERIALS:

1. *If no overhead projector is available, square shapes*
2. *Tiles*
3. *Individual blackboards*
4. *Unlined paper*

The following two lessons focus student attention specifically on the variety of number combinations or "number facts" in addition and subtraction that are associated with a given sum. The students learn many different combinations of numbers will produce the same answer.

Teacher: Take out eight tiles. Before, when you worked with tiles, I asked you to either add or subtract to get an answer. This time, I'll give you the answer and you can tell me what the problem might have been.
The answer we'll start with is "eight." What was the problem?

The teacher's question is meant to lead the class into a general discussion of what problems have an answer of eight. However, the class may need a more specific explanation of what is wanted.

Teacher: If I divide my eight tiles into two equal groups, what problem did I start with?
Student: Four and four.
Teacher: That's one problem. Can you tell me another that might have eight tiles as its answer?

The problems need not be limited to addition or subtraction. The students' imaginations should be the only limit.

When the teacher's question is understood, the students begin devising and recording their own problems. Once they feel they have exhausted the possibilities for eight, they may choose another number.

While constructing problems to go with their numbers, students may explore the answers to the following questions:

Are there more problems that can be made up for eight than for seven? How about six?
How many problems can you make up with zero as an answer? Do more or less problems have zero as an answer than eight? How do you know?
What is the highest number of problems you can make up for nine?
What would be the most if you only used addition? Subtraction?
What is the most unusual problem you can make up?
What is the longest problem you can make up? The shortest?

A student may decide that $0 + 8$ is different from $0 + 0 + 0 + 8$. This permits the student to generate an endless number of problems simply by adding more zeros. Whether $0 + 8$ is different from $0 + 0 + 0 + 8$ is a matter for the class to discuss and decide. Once discussed, the class resolves the issue by voting. Whatever the class decides, will be the rule for resolving all similar questions. The rule thus created stays in effect until the class decides to change it.

LESSON 4-11

NUMBER FACTS WITH TILES

PURPOSE:

To provide experience in developing specific problems to match a given answer

MATERIALS:

1. *Tiles*
2. *Individual blackboards*
3. *Unlined paper*

Teacher: Take out eight tiles. This time I want you to find problems that have the answer eight, but only use *two* groups of tiles.

When you think of a problem, tell me and I'll write it down. Be sure to check the overhead to see if the problem you have made has already been recorded.

Student: Three plus five.

Student: One and seven.

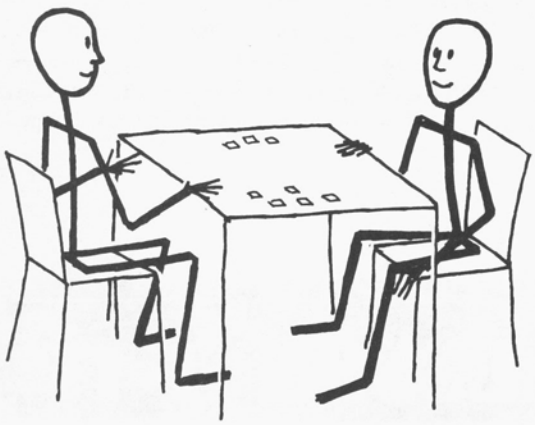
Student: Four and four.

Student: Two plus six.

Student: Five and three.

Student: Five plus three is already up there. It's the same as three plus five.

The teacher has an opinion about whether five plus three is the same as three plus five, but it isn't shared with the class. The students must resolve this issue to their own satisfaction through discussion then vote. If the students decide the two ways are the same, they will find this decision challenged when they reach coordinate graphing. If the class decides the ways are different, then how different are they? Shall the students now count problems such as the one shown with tiles in the figure below as a different problem depending on one's physical location? A student sitting in front of the problem sees five plus three. A student facing the problem from the back sees three plus five. However the vote comes out, events in the future will cause them to rethink the issue. They may never be completely satisfied with their decision.



Even the most valid rules or axioms of mathematics are only the product of our ancestors' ability to decide what seemed reasonable and to convince their fellows that what seemed reasonable to a few should be accepted as reasonable for all. Our students' understanding of the rules of mathematics is enhanced if they can decide for themselves what is reasonable and live with their collective decision.

Once the students have told the teacher as many ways as they can think of to get eight as an answer using two groups of tiles, they pick a new starting number, such as seven or nine, and continue as before. Whatever rules the class has chosen when finding ways to make eight are binding on each new number attempted.

LESSON 4-12

ADDITION WITH CUBES

PURPOSE:

To create and record simple addition problems with cubes

MATERIALS:

1. *If no overhead projector is available, square shapes*
2. *Unifix cubes*
3. *Individual blackboards*
4. *Unlined paper*

This lesson combines the activities presented in Lessons 4–1 and 4–2. The only substantial difference is that this lesson substitutes cubes for tiles.

Two reasons exist for having the students, using a new material, repeat lessons they have recently completed. First, using a new material with an old lesson permits the teacher to reteach those students who need additional experience. A new material provides the opportunity for remediation and review without attaching any stigma to the process.

The second reason is to counteract a common difficulty when only one material is used to convey a concept to students. They have seen the patterns created by numbers with tiles, so they know what is true for tiles is true for tiles. For students to learn that the patterns they have seen apply to a variety of situations, they need to be presented other examples which lead to the same conclusion. If they are not given the opportunity to transfer what they have learned from one situation to another, in many cases, no such transfer takes place.

For some students, the activities with cubes may seem like a totally new exercise; others may comment that the cube patterns are the same as the ones they found for

tiles. Both groups may be given the same assignment: find which patterns for cubes are the same as for tiles and which patterns are different.

LESSON 4-13

SUBTRACTION WITH CUBES

PURPOSE:

To create and record simple subtraction problems with cubes

MATERIALS:

1. *If no overhead projector is available, square shapes*
2. *Unifix cubes*
3. *Individual blackboards*
4. *Unlined paper*

The activities for this lesson combine the activities presented in Lessons 4-3 and 4-4, only substituting cubes for tiles.

LESSON 4-14

ADDITION AND SUBTRACTION WITH CUBES

PURPOSE:

To provide practice in addition and subtraction while searching for patterns in number

MATERIALS:

1. *Unifix cubes*
2. *Individual blackboards*
3. *Lined paper*

This lesson combines the activities presented in Lessons 4-8 and 4-9. The same procedures are used except that the students may choose to explore either addition or subtraction patterns, or both, using cubes instead of tiles.

The students now have a framework for creating and checking their own addition and subtraction problems. They have encountered a variety of situations in which they have been encouraged to look for patterns in numbers. This search has been conducted not only within experiences, but also between experiences. The activities in the next chapter capitalize on the students' growing ability to see patterns in numbers.