CHAPTER 9
Advanced Multiplication

Prerequisite chapters:

Chapters 5 and 8

MATERIALS

For overhead projector:

Transparencies	Special multiplication matrix	Worksheet 15
	Boxes, two squares by two squares	Worksheet 16
Acetate squares in five different colors		Materials chapter, page 297
Washable color marking pens		Materials chapter, page 297

If no overhead projector is available:

Make charts in place of transparencies	Materials chapter, page 294
Strips and squares of paper in five different colors	Materials chapter, page 297

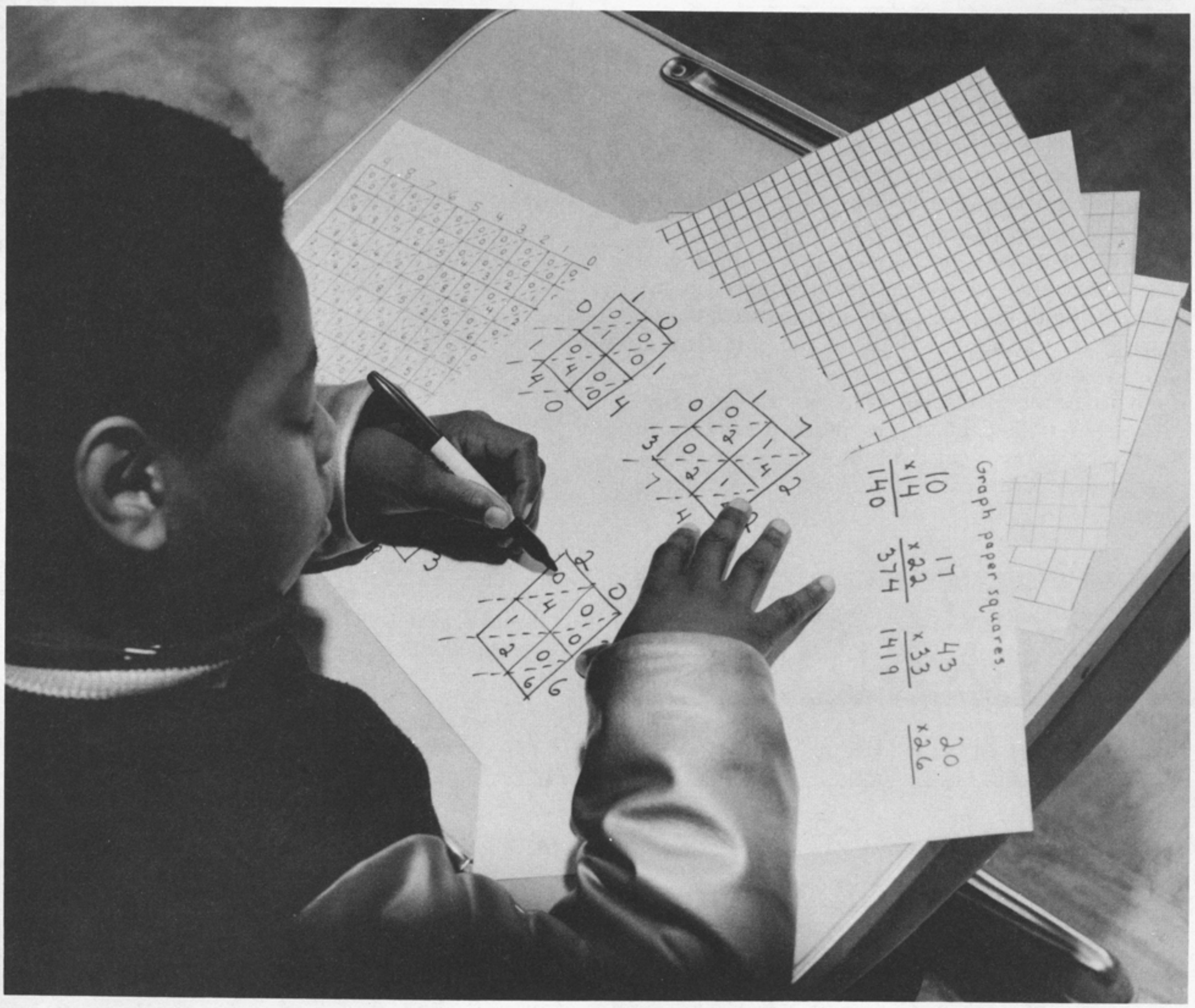
Student materials:

Dittos	Special multiplication matrix	
	Boxes, two squares by two squares	
Paper squares or chips in five different colors		Materials chapter, page 297
Individual blackboards		Materials chapter, page 294
Spelling notebooks		Materials chapter, page 296

Crayons

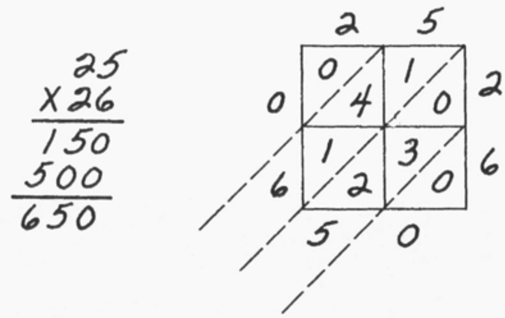
Lined and unlined paper

Assorted materials to accompany questions on page 119 of Lesson 9-4 and page 125 of Lesson 9-9.



In Chapter 5, students learned to use materials such as tiles to calculate answers to multiplication problems whose products were less than 100. As problems increase in size, use of materials soon becomes impractical.

Traditionally, an abstract system of long multiplication involving the distributive process has been taught. Research has shown, however, that the lattice system of multiplication allows students to compute multi-digit multiplication problems in significantly less time and with greater accuracy than is possible using the distributive method. (Frank George Hughes, *A Comparison of Two Methods of Teaching Multidigit Multiplication* [University of Tennessee, 1973]. DAI 34A: 2460–2461; 1973.) The lattice method produces the same level of understanding as the distributive method but is easier to teach, faster to use, and less prone to error. Consequently, this method is the one presented in this chapter.



Students who have difficulty reasoning abstractly with numbers are frequently unable to grasp the numerical logic behind either the distributive or the lattice approach to long multiplication. Knowledge of *why* an abstract system of producing answers works is not as important as the knowledge that it *does* work. For this reason, answers to the initial problems students work using a lattice are checked against materials, usually the chips on the chip trading boards.

The initial lessons that follow acquaint students with using their chip trading boards to produce answers to multiplication problems. Once this use is understood, the boards are used to check answers obtained through lattice multiplication.

LESSON 9-1

GROUPING WITH CHIPS

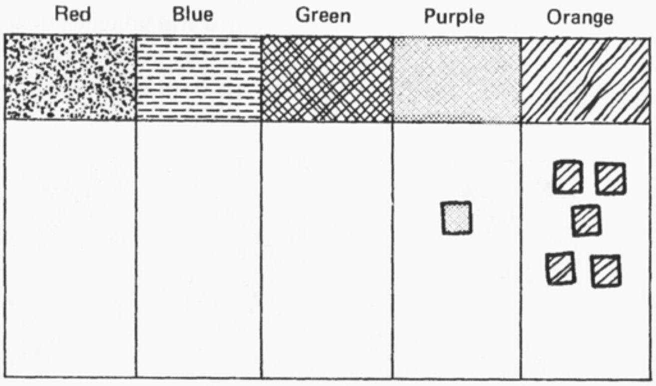
PURPOSE:

To assign place values to piles of paper chips in preparation for learning to compute answers to multiplication problems on chip trading boards

MATERIALS:

1. *Acetate squares or squares of colored paper in five different colors*
2. *Marking pens with washable ink, or colored strips of paper in five different colors*
3. *Paper chips in five different colors*
4. *Crayons*
5. *Paper, lined and unlined*
6. *Individual blackboards*

Teacher: How many orange chips do I have?
Student: Fifteen.
Teacher: Take out a pile of fifteen orange chips and trade them on your chip trading board.
Student: How are we grouping?
Teacher: Into groups of ten.



When the students have completed the exchange, they record the number of chips on their blackboards.

Teacher: I can see the numbers you have on your blackboards. Would someone please read me what you got?
Student: One, five.
Teacher: One, five what?
Student: One purple and five orange.
Teacher: Okay. This time I want you to keep track of the whole problem on your blackboards. Draw a line down the middle, and on the left side, write the number of oranges you start with. On the right side write the number for the chips on your trading board when you get through making all the trades you can. The problem you just did would be recorded like this.

Start	Trade in
15	00015

Take a pile of 17 orange chips, trade them on your chip trading board, and then record the chips you started with and the chips you ended with on your blackboards.

The teacher tells how many orange chips to start with until they understand the recording process. Each student then chooses a starting pile of orange chips, trades them on the chip trading board, and records the appropriate numbers on lined paper.

As the students work, the teacher asks the class to think about the following questions:

Can you see a pattern between the numbers in the starting pile and the numbers after you've made all possible exchanges?
Could you use the pattern to help you predict what numbers you'll get?
Will this pattern still work if you start with bigger piles of oranges?
How big a pile of oranges would you need for the pattern to stop working?

LESSON 9-2

MULTIPLICATION WITH CHIPS

PURPOSE:

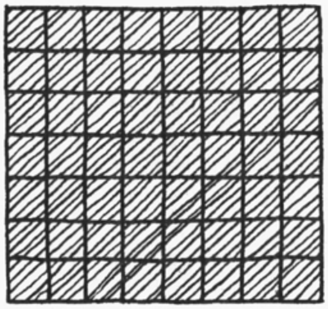
To assign place values to paper chips formed into rectangles in preparation for computing answers to multiplication problems on chip trading boards

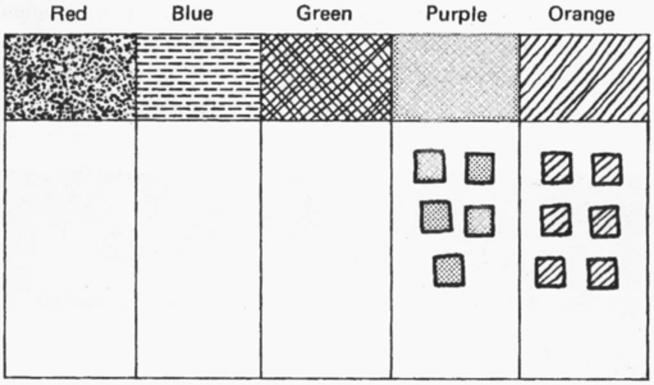
MATERIALS:

1. *Acetate squares or squares of colored paper in five different colors*
2. *Marking pens with washable ink, or colored strips of paper in five different colors*
3. *Paper chips in five different colors*
4. *Crayons*
5. *Paper, lined and unlined*
6. *Individual blackboards*

The activities for this lesson are essentially the same as those in the preceding one. The difference is in how the piles of orange chips are formed.

Teacher: Using orange chips, make a rectangle with seven rows and eight columns. Take the orange chips from the rectangle and use them to make as many exchanges as you can on your chip trading board. When you finish, write the number of chips you have on your blackboards.





For the second rectangle, the students use their blackboards to record the number of rows and columns and the final number of chips. The appropriate numbers for the first rectangle would be recorded as shown in this figure.

Rows	Columns	Chips
7	8	00056

When students understand the process, they construct rectangles of any size they wish and record the appropriate numbers on lined paper.

The teacher poses these questions:

Can you see a relationship between the numbers for the rows and columns and the numbers you get after you've exchanged all the orange chips you can?
Can you use this relationship to help you predict what numbers you might get in your chips column, once you know how many rows and columns are in your rectangle?
Does this relationship still work if you start with bigger rectangles?
How big a rectangle would you need to make before you could not find a relationship between the three numbers?

In the following lessons, students will be asked to exchange large numbers of orange chips for chips with other values. The number of orange chips is the same number obtained when all the appropriate chip exchanges have been made. Thirty-two orange chips converts to three purples and two oranges because thirty-two ones is the same as three tens and two ones.

For students to make sense out of the activities that follow, they must either recognize the relationship between the total number of orange chips and the chips for which the orange may be exchanged, or be able to make the appropriate exchanges on their boards. These first two lessons allow students to discover these patterns. If they cannot, they at least can make the necessary exchanges on their boards.

The activities also direct the students' attention to the relationship between the numbers on their chip trading boards and the numbers they recorded earlier on their multiplication tables. Rectangles constructed from orange chips provide students with their first evidence that answers to multiplication problems can be displayed on chip trading boards. The following lesson expands this notion.

LESSON 9-3

MULTIPLICATION WITH CHIPS

PURPOSE:

To relate multiplication with chips to real problems

MATERIALS:

1. *Acetate squares or squares of colored paper in five different colors*
2. *Marking pens with washable ink, or colored strips of paper in five different colors*
3. *Paper chips in five different colors*
4. *Crayons*
5. *Individual blackboards*

Proof that an abstract system of multiplication produces correct answers for concrete situations is best obtained by matching the system's answers with those found for the concrete situations. The answers for the problem situations in this lesson are found through the use of chip trading boards, and are eventually compared with answers obtained through lattice multiplication.

Teacher: How many classrooms do we have in our school?
Student: Twenty-four.
Teacher: How many people in our room today?
Student: Thirty-one.
Teacher: I will assume there are 31 people in each classroom. How can we find out how many people there are altogether in our school?
Student: Count them.
Student: Add up 31, 24 times.
Student: Ask the office to tell us.
Student: Take 24 times 31.
Teacher: There are many ways we might find out. Today, though, I'll show you how to find the answer with your chip trading board.
I will select 24 students, each to represent a different classroom. How many students did we say each classroom has?
Student: Thirty-one.
Teacher: Then each of you who is representing a classroom take out 31 orange chips. Can you tell me why?
Student: Because the chips represent the students in each class.
Teacher: Now, how can we use the chips to find out how many students are in our whole school?
Student: We could count them.
Teacher: Okay. Each person who is representing a classroom, count your orange chips.
Student: No! We already know each has 31 chips. We have to count everybody's chips!
Teacher: I could have each of you bring your orange chips up and we could put them in a pile and start counting. But, I won't. I'd like you to trade in as many of your orange chips as you can on your chip trading board. If you do that, we won't have as many chips to count. What did you get when you traded in your 31 orange chips?
Student: Three purples and an orange.
Teacher: Did you all get the same thing?
Student: Yes.
Teacher: Okay. One at a time, I want each of you who represents a classroom to come up here and bring your three purples and an orange. I'll take your chips and add them together on my overhead chip trading board.

The teacher collects all the chips, adding each new group of three purples and an orange, being careful to make all the appropriate trades.

Teacher: The answer is seven greens, four purples, and four oranges. What were we trying to find out?
Student: How many oranges we all had altogether?
Teacher: True. What were the orange chips representing?
Student: Children in each classroom.
Teacher: How many children did we say were in each classroom?
Student: Thirty-one.
Teacher: And how many classrooms?
Student: Twenty-four.
Teacher: So, how many children did we find were in the whole school?
Student: What do you mean?
Teacher: How many orange chips did we have when we added them all together?
Student: I don't know.
Teacher: What is the answer we got?
Student: Seven green, four purple, and four orange.
Teacher: If you took seven greens, four purples, and four oranges and changed them all into orange chips, how many orange chips would you have?
Student: Seven hundred, forty-four oranges.

Teacher: What did each orange stand for?
Student: A child in a classroom.
Teacher: So, how many children are in all the classrooms.
Student: Seven hundred and forty-four children.

Knowing what the answer means is equally important as having the correct numbers. If 21 children each had 20 fingers and toes, how many fingers and toes do the children have altogether? The numerical answer is 420, but 420 what? Students? Toes? Fingers?

A student might answer this problem as follows:

Student: Twenty-one plus 20 is 41. The answer is 41.
Teacher: Forty-one what?
Student: Forty-one!

A student who knows to look for the total number of fingers and toes has a better chance of distinguishing a reasonable answer from one that may be absurd.

The teacher guides the class through as many examples of real life problems as time permits.

There are thirty-one students in class. If each one ate forty pancakes, how many pancakes would they eat altogether?
If each person in class had twenty cousins, how many cousins would there be altogether?
If everyone in our class had forty-five dollars, how rich would we be?

LESSON 9-4

MULTIPLICATION WITH CHIPS AND OTHER MATERIALS

PURPOSE:

To practice finding answers to two-digit times two-digit multiplication problems using chips and other materials

MATERIALS:

1. *Paper squares or chips in five different colors*
2. *Crayons*
3. *Individual blackboards*
4. *Spelling notebooks*
5. *Paper, lined or unlined*
6. *Assorted materials, depending on the teacher's questions*

This lesson continues the activities of the preceding one. However, the students themselves find the answers, using, if they wish, their chip trading boards or any other method they think helpful.

The teacher presents about eight or ten problem situations for the class to discuss. When they have all been discussed, the teacher reviews each quickly. Each student then chooses a problem to work. The students may work alone or in teams of twos or threes. No limit is placed on the number of students or groups of students who may work any one problem.

Students record the answer on a piece of paper and write a brief description of what the numbers represent. For example: 18 rows, 24 columns, 432 tiles altogether. The brief descriptions help the teacher know which problem was worked.

Examples of the kinds of problem situations the teacher might present to the class for discussion and solution are among the following:

How many squares on this sheet of graph paper?
How many Unifix cubes or tiles would it take to cover your desk with rows all of the same length?
If you took all the beans in this jar and put the same number in each of these cups, how many beans would be in the cups altogether? Don't count any leftover beans.
I have typed a whole page of *x*'s on this paper. How many *x*'s are there altogether?
How many chips would there be altogether if each person in class dropped 19 orange chips into a box?
How many words would be written if each person in class wrote 15 words on a page?
How many books would be in this room if each of us had 25 books?

Possible questions are endless. The only controlling factors for the present lesson are (1) the questions should produce multiplication problems, and, (2) the problems created should involve multiplying two-digit numbers times two-digit numbers.

LESSON 9-5

MULTIPLICATION WITH CHIPS AND OTHER MATERIALS

PURPOSE:

To practice finding answers to two-digit times two-digit multiplication problems using chips and other materials

MATERIALS:

1. *Paper squares or chips in five different colors*
2. *Crayons*
3. *Individual blackboards*
4. *Spelling notebooks*
5. *Paper, lined or unlined*
6. *Assorted materials, depending on the teacher's questions*
7. *The students' recorded work from Lesson 9-4*

This lesson continues the activities of Lesson 9-4. The students choose a second problem to solve from the ones presented in the previous lesson. They complete as many problems as time permits.

LESSON 9-6

MULTIPLICATION WITH CHIPS AND OTHER MATERIALS

PURPOSE:

To check the answers found for earlier multiplication problems using chips and other materials

MATERIALS:

1. *Paper squares or chips in five different colors*
2. *Crayons*
3. *Individual blackboards*
4. *Assorted materials, depending on the teacher's questions in Lesson 9-4*

Teacher: I have looked through the papers you handed in yesterday and the day before. I am a little confused about a few of the problems. Some people started with exactly the same numbers, yet found different answers and I don't know which is right. Is it possible to start with the same numbers, end up with two different answers, and have both answers be right?
Today, I want you to rework those problems and find out if there can be more than one right answer. I'll write the numbers for the problems on the overhead, and the different answers. Pick one of the problems and work it out. When you get an answer, tell me which of the answers on the overhead you agree with. If you get an answer that's different, tell me that, too. I'll write a couple of reminding words by the numbers so you can remember which numbers go with which materials.

18 rows	24 columns	444 or 432 tiles altogether
23 cups	15 beans in each cup	345 or 339 beans altogether
⋮	⋮	⋮

Students are capable of checking the answers to problems produced from real events. If we tell them which answer is right, we deny them the opportunity to exercise their own thinking power to resolve the issue.

Each time a student finds an answer, the teacher writes that answer after the problem. The students continue finding answers until about ten minutes remain in the period. The teacher then discusses the answer.

Teacher: Look at the first problem on the overhead. What is the most common answer people found for how many tiles altogether?
Student: Four hundred and thirty-two.
Teacher: I will accept 432 as the answer. I'll call it the *class answer* since it's the answer the class found most often.

When a class answer has been established for each problem, the teacher and the class discuss these questions:

If the class answer is correct, why wasn't it the only answer people found?
Is it possible some of the other answers are right, too?
Can there be problems with more than one right answer?
Can there be problems with no right answer?
Do you think it's all right to say an answer is the correct answer just because more people found that answer than any other?

All the problems worked by the students in this lesson and the two preceding ones will be used again in the two that follow.

LESSON 9-7

LATTICE MULTIPLICATION

PURPOSE:

To learn an abstract system for multiplying numbers with products greater than 100

MATERIALS:

1. *Special multiplication table for multiplying with boxes on an overhead transparency or on a large tagboard*
2. *Dittoed copies of the special multiplication table*
3. *Record sheets from Lessons 9-4 and 9-5 for reference*
4. *Individual blackboards*

Teacher: Today I will show you how to solve multiplication problems too big to find on a multiplication table. You will learn a way that you probably have never seen before. It's not the same system you've been taught in other grades.

This system was thought up a long time ago by a man named John Napier, who lived in Scotland and was good in mathematics. His way of working multiplication problems is easier to use than any other way I know.

I will use the problem we worked a few days ago for finding how many people were in our whole school. How many students did we say were in each classroom?

Student: Thirty-one.

Teacher: How many classrooms?

Student: Twenty-four.

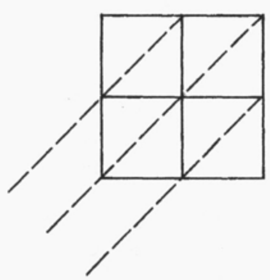
Teacher: Can anyone remember how many students we found in the whole school? I think it might have been 744.

Student: Yes, it was.

Teacher: If I write 31 times 24 like a regular multiplication problem, it would look like this.

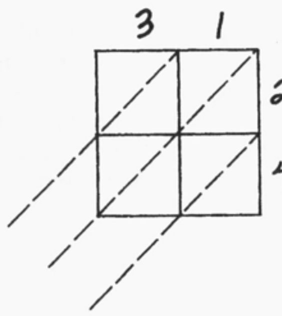
$$\begin{array}{r} 31 \\ \times\ 24 \\ \hline \end{array}$$

Now, I'll show you the new method. I call this *multiplying with boxes,* because you have to write the numbers for the problem around a box like this.



I'll write the number that tells how many people are in each room at the top. Notice I wrote one number over the top of each square. Down the right side, I'll write the numbers that tell how many classrooms there are. Each number goes opposite one of the squares on the side.

I could have written the number of people in each room on the right side and the number of classrooms at the top— it doesn't make any difference.

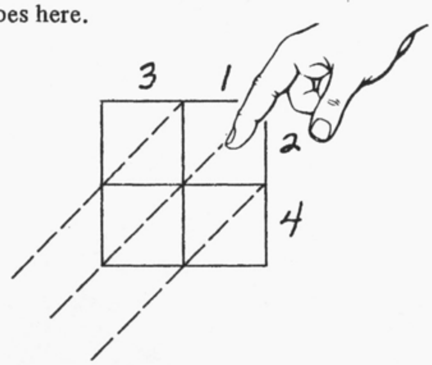


The numbers that go inside the little boxes come from a special multiplication table.

9	8	7	6	5	4	3	2	1	0	
0/0	0/0	0/0	0/0	0/0	0/0	0/0	0/0	0/0	0/0	0
0/9	0/8	0/7	0/6	0/5	0/4	0/3	0/2	0/1	0/0	1
1/8	1/6	1/4	1/2	1/0	0/8	0/6	0/4	0/2	0/0	2
2/7	2/4	2/1	1/8	1/5	1/2	0/9	0/6	0/3	0/0	3
3/6	3/2	2/8	2/4	2/0	1/6	1/2	0/8	0/4	0/0	4
4/5	4/0	3/5	3/0	2/5	2/0	1/5	1/0	0/5	0/0	5
5/4	4/8	4/2	3/6	3/0	2/4	1/8	1/2	0/6	0/0	6
6/3	5/6	4/9	4/2	3/5	2/8	2/1	1/4	0/7	0/0	7
7/2	6/4	5/6	4/8	4/0	3/2	2/4	1/6	0/8	0/0	8
8/1	7/2	6/3	5/4	4/5	3/6	2/7	1/8	0/9	0/0	9

The special multiplication table is a regular multiplication matrix that has been modified to permit the maximum number of students to comprehend the process of taking numbers from the table and placing them in correct positions in the lattice. For most students, a regular multiplication matrix would suffice. The special table has proved extremely advantageous, however, for those with learning difficulties. To avoid singling them out, *all* students use the special table for lattice multiplication.

Teacher: Use the special multiplication table to predict what goes here.



Although it is unlikely the students will know what number goes in the box, allowing them to anticipate an answer heightens their interest.

If a student says two goes in the box, the teacher asks that student to explain to the class where the two came from. If no one can guess, the teacher explains the use of the special matrix.

Teacher: What number is in the row of numbers directly above the square I pointed to?

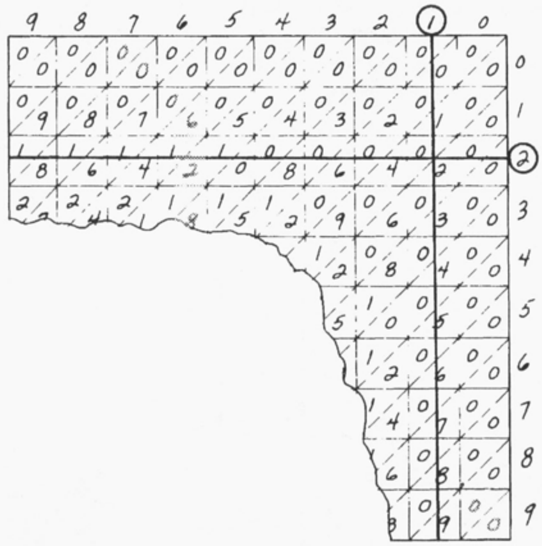
Student: One.

Teacher: Okay, find the one on the top row of your matrix and draw a very light circle around it. Now draw a light line straight down from the one like this.

What number is in the column of numbers directly to the right of the square I pointed to?

Student: Two.

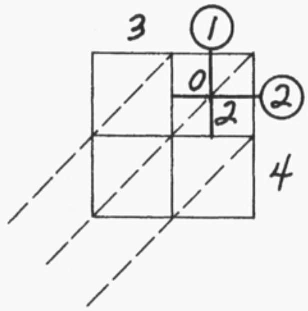
Teacher: Okay, find the two in the column of numbers down the right side of your matrix and draw a light circle around it. Now draw a light line straight across from the two, like this.



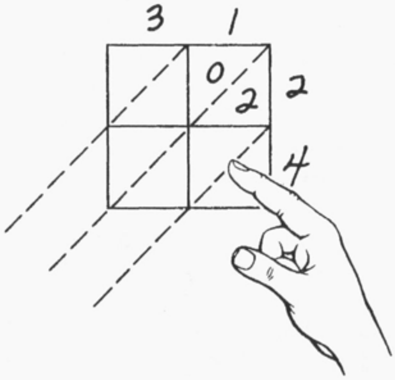
Do you see where the two lines cross?

Student: Yes.

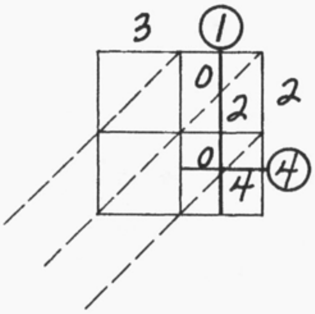
Teacher: The numbers in the square where the two lines cross are the numbers that go in the box I pointed to.



What goes here?

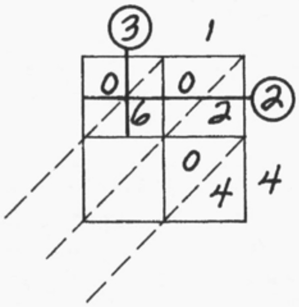


Student: A zero and a four.

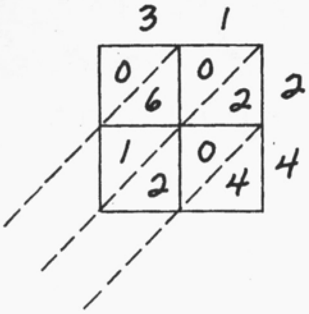


Teacher: And here?

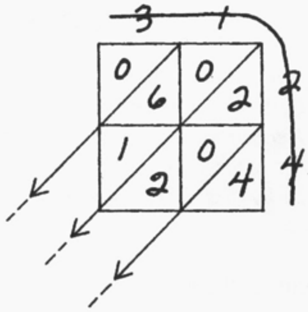
Student: A zero and a six.



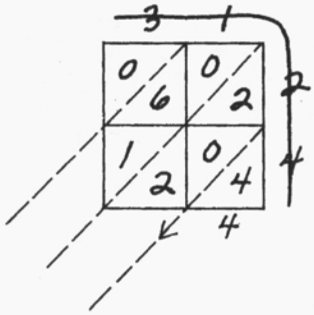
Teacher: And in the last empty space?
Student: One and two.



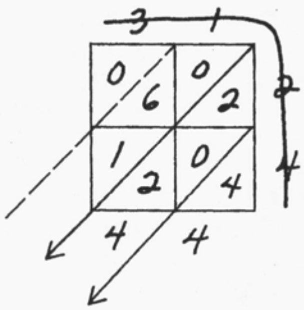
Teacher: Okay. Now, to get the answer to the multiplication problem, I add the numbers next to the dotted lines. I'm only going to add the numbers that are *inside* the little squares. I've already used the numbers on the outside to help me get the inside numbers, so I'll cross them off.



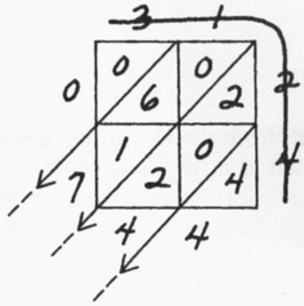
What's four plus nothing else?
Student: Four.



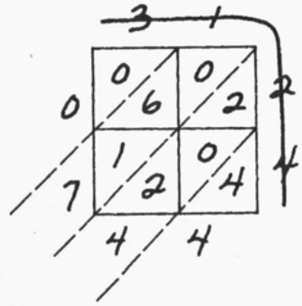
Teacher: What's two plus zero plus two?
Student: Four.



Teacher: What's zero plus six plus one?
Student: Seven.



Teacher: What's the answer for the last space?
Student: Zero.



Teacher: That makes the answer for this problem 0744. Is 0744 the same answer we got when we worked the problem on our chip trading boards?

The teacher works another example for multiplying a two-digit number by a second two-digit number, drawn from the problems for which the class found answers in the previous three lessons.

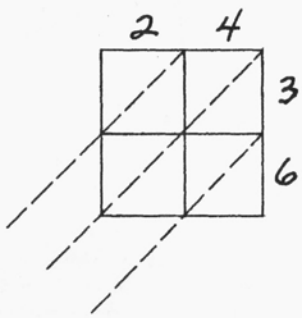
For the third example, the students should direct the teacher in each step.

Teacher: When we made a giant rectangle of cubes with 24 columns and 36 rows our problem looked like this.

$$\begin{array}{r} 24 \\ \times\ 36 \\ \hline \end{array}$$

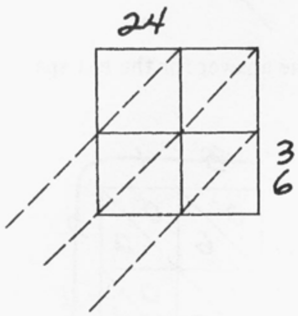
The answer for how many cubes altogether was 864. Can you tell me what to do using the boxes so I can get the same answer?

Student: Put the 24 along the outside of the top of the boxes and the 36 along the outside on the right of the boxes.

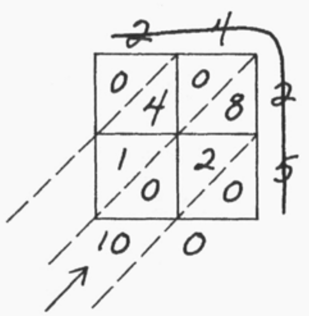


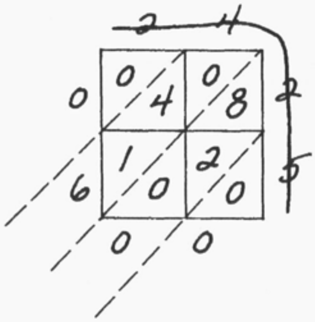
Teacher: Like this?
Student: Yes.
Teacher: Now what?

The 24 and the 36 from the above figure could equally well have been written as in the figure below. The better the students become at instructing the teacher, the more exacting the teacher may require the instructions to be.



The teacher will need to clarify one potential area of confusion for the students when it arises. The problem in the figure below is essentially the same as other problems the students might have seen worked, except the second diagonal adds to ten. This means two numbers occupy a single space. In none of the spaces in the little squares for the box problems, has there been more than a single digit—each is separated from all others by either solid or dotted lines. The rules for multiplying on lattices require this separation of digits. If there is a diagonal that produces a two-digit total, one of the digits must be carried over to the next diagonal and added to that diagonal's numbers; the digit *closest* to the next diagonal is the one to be carried over. For this example, the one would go and the zero would stay.





For the remainder of the lesson, the students instruct the teacher in finding answers to box multiplication problems, but do not yet work any problems on their own.

LESSON 9-8

LATTICE MULTIPLICATION

PURPOSE:

To practice an abstract system of multiplying numbers with products greater than 100

MATERIALS:

1. *Special multiplication table for multiplying with boxes on a transparency or a large tagboard*
2. *Boxes two squares by two squares on a transparency or a large tagboard*
3. *Dittoed copies of the special multiplication table*
4. *Dittoed copies of the boxes*
5. *Record sheets from Lessons 9-4 and 9-5, for reference*
6. *Individual blackboards*

This lesson continues the activities of the previous lesson, beginning with a review of the procedures for multiplying with boxes.

Once the review is complete, the students are ready to begin working problems on their own. They are given a dittoed page with several blank multiplication boxes on it for recording their work. Learning how to draw the boxes and put in the dotted lines is a perceptual activity that should not be taught at the same time the students are mastering the steps for producing multiplication answers using the boxes.

Teacher: I will write some problems on the overhead; they are all ones you solved earlier using materials or your chip trading boards. I want you to find answers using the boxes.

I know the few box problems we have worked so far have given us the same answers we got before with materials. I am not sure this will happen for every problem. Today I want you to find examples for the boxes that give you different answers than we got with the materials.

I have saved all the papers on which you recorded the numbers and answers for the materials problems. If the answer on the paper wasn't a class answer, I wrote it there also.

I will put one paper for each problem on the front counter. When you finish, come up to the front counter and check your answer against the paper. If the answer you found using the boxes doesn't agree with the class answer on the paper, then you may have found an example of a box problem that doesn't give the same answer as a materials problem. If this happens, find one other person in class who has worked your problem using the boxes. If both of you have the same answer bring your box problems to me and we'll have the rest of the class verify your discovery.

The teacher writes problems from Lessons 9-4 and 9-5 on the overhead, telling the class what the numbers refer to, and writing a few key words next to the numerals. The key words help the students pick out the appropriate recording sheet for checking the answers to their box problems.

40	Graph paper	19	Chips	15	Cups
x 35	Squares	X 31	Students	X 17	Beans

Students who do not get the class answer to a problem have made a mistake; however, the teacher does not correct it. The first chance at correction comes when the student searches for a fellow classmate with the same answer. If two students who have made the same mistake find each other, the second chance at correction occurs when the problem is presented to the class. The class as a whole is a competent problem-solver.

LESSON 9-9

LATTICE MULTIPLICATION

PURPOSE:

To practice an abstract system of multiplying numbers with products greater than 100

MATERIALS:

1. *Boxes, two squares by two squares on a transparency or a large tagboard*
2. *Dittoed copies of the special multiplication table*
3. *Dittoed copies of the boxes*
4. *Individual blackboards*
5. *Assorted materials, depending on the teacher's questions*

In the previous two lessons, the students used boxes to reproduce answers they had already found using materials. Now, the process is reversed: they first find answers using boxes then in the next lesson verify their results with materials.

The teacher presents problems involving materials or the chip trading boards. Groups of students then decide the appropriate numbers for the teacher to place on the boxes at the overhead, and copy them onto their dittoed sheets to work out the answers.

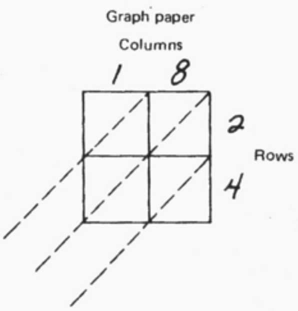
Before groups of students examine any material or problem situation for the numbers to place around the boxes on the overhead, the class as a whole discusses what numbers are to be looked for. The problem situations that the teacher presents are essentially the same as those in Lesson 9-4, but the numbers are not. The situations are kept the same so the class will have little difficulty determining what numbers each group of students is to find.

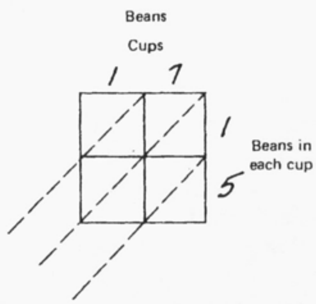
Teacher: Brenda and Russell, in a moment I want you to take this piece of graph paper and find the numbers I should put around the boxes to find how many squares there are altogether. Don't tell me how many there are, just the numbers I need to find out.
Before they begin work, can anyone tell me what numbers they will give me?
Student: How many rows and how many columns.
Teacher: Okay . . . I need to know how many rows and how many columns on this paper so I can find out how many squares altogether using the boxes.

Additional examples of old situations that will lead to new numbers are:

I want three people to cover my desk with Unifix cubes. What numbers must they give us so we can find how many Unifix cubes they used altogether?
Take the beans from this jar, put an equal amount into each of these cups. What numbers do we need to use the boxes to find out how many beans are in the cups altogether?
Here is a whole page of *A*'s. What numbers do we need to use the boxes to find out how many *A*'s are on this page altogether?
Each of you drop 25 chips into this box. What numbers do I need to find how many chips I will have altogether?

The teacher asks the students to provide numbers for ten or fifteen different problem situations. So the numbers from each situation may be identified, the teacher also writes one or two key words next to them. Each new situation is patterned after a familiar experience with materials so the students can compare.





The students use their boxes to work out the answers. Then, the teacher points to the problems on the overhead one at a time and asks them to write their answers on blackboards. The teacher then selects the most frequent answer, as the class answer.

These box problems and their answers form the basis of the next lesson.

LESSON 9-10

MULTIPLICATION WITH CHIPS AND OTHER MATERIALS

PURPOSE:

To verify results of multiplying with boxes using materials

MATERIALS:

1. *Assorted materials, depending on the teachers questions in Lesson 9-9*
2. *Paper, lined or unlined*
3. *Spelling notebooks*
4. *Chips and chip trading boards as needed*
5. *Box problems with answers recorded by the teacher from Lesson 9-9*

Teacher: I've put the problems you found answers for yesterday on the overhead. Today I want you to use the materials from which you determined the starting numbers for each problem to find the answer.

Each group will work on the same problem as yesterday to find the number for how many altogether.

What would I want Russell and Brenda to find out about their graph paper?

Student: How many squares are on the page altogether.

Teacher: Okay, every other group will find out how many beans, or chips, or books they had altogether, too. When your group has found the answer, write it on a piece of paper and bring it to me, then select another problem. I would like at least two different groups to work each problem so we can see if the answer comes out the same.

Students may think they have found different answers with the materials than with the boxes for some problems. If one group finds a difference, a second group works that problem. If they find a difference, too, even though it may not be the same difference, the problem is presented to the whole class. The students try to determine if this is an exception to the fact that multiplying with boxes produces the same answer as does actually counting out the materials.

When groups finish their own problem, they are assigned the class problem if there is available material. Each answer is listed on the overhead, until the inevitable conclusion is reached that the problem is *not* an exception.

If one group finds a difference between the box answer and the materials answer, and a second group does not, the two groups must decide if that problem is counted as an example of a materials answer different from the class answer found for boxes.

The students continue checking each box answer against the material it relates to. Each answer is written on the overhead next to the appropriate problem. By the end of the period there should be a substantial display of evidence that multiplying with boxes produces the same answers as does counting.

LESSON 9-11

LATTICE MULTIPLICATION

PURPOSE:

To learn how to draw boxes; to practice box multiplication techniques while determining answers to selected multiplication problems

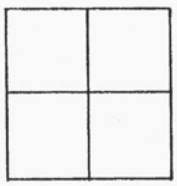
MATERIALS:

1. *Individual blackboards*
2. *Unlined paper*
3. *Dittoed copies of the special multiplication matrix*

Some students' specific learning difficulties make it necessary that they be taught in detail how to draw the boxes. So they are not singled out, the teacher teaches everyone how to draw boxes.

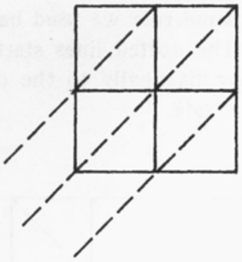
Teacher: Today I will show you how to draw the boxes you have already been using. As I draw mine on the overhead, copy it on your blackboards.

First, draw a box. Now, divide it in half from top to bottom and again, from side to side. How many little squares are inside the box?

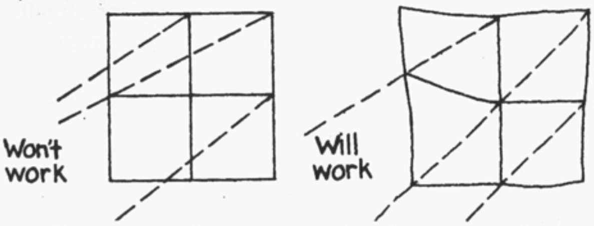


Student: Four.

Teacher: Putting in the dotted lines is the most difficult part. Each goes diagonally from one corner of each little square across that square to the other corner and then outside.



It is important to stress that dotted lines go from corner to corner of each little square, or some students will start a dotted line in one corner and end up no place in particular. In addition, dotted lines in squares not all equal in size do not always end up straight. This has no effect on the boxes ability to produce an answer, as long as each individual square is bisected by the dotted line.



The teacher demonstrates drawing a box on the overhead. For the box the students draw, the teacher describes the procedure but does not accompany the words with a drawing. The students collectively tell the teacher how to draw the third box on the overhead. The students draw the fourth box on their blackboards with no instructions from the teacher, although a model is drawn on the overhead.

When the students can draw their own boxes, the teacher gives them a new kind of multiplication problem.

Teacher: Watch as I write this series of multiplication problems. What do you think the next problem will be?

$$\begin{array}{r}25\\ \times 25\\ \hline\end{array}\quad \begin{array}{r}24\\ \times 26\\ \hline\end{array}\quad \begin{array}{r}23\\ \times 27\\ \hline\end{array}\quad \begin{array}{r}22\\ \times 28\\ \hline\end{array}$$

Student: Twenty-one times twenty-nine.

Teacher: I want you to multiply each problem using the boxes. You can tell what the next one will be by looking at the problems already written on the overhead. As you work, search the answers for a pattern to help you predict the answers to the next problem you do.

Students may or may not discover the patterns that exist in answers to multiplication problems presented in a sequence. If they do, they may share their discoveries with their classmates.

When one series of problems has been explored the teacher writes another. The numbers in the problems advance or decline in a predictable manner. The students may, if they wish, attempt to construct their own sets of problems to explore.

$$\begin{array}{r}20\\ \times 20\\ \hline\end{array}\quad \begin{array}{r}19\\ \times 19\\ \hline\end{array}\quad \begin{array}{r}18\\ \times 18\\ \hline\end{array}\quad \begin{array}{r}17\\ \times 17\\ \hline\end{array}\ \ldots$$

LESSON 9-12

LATTICE MULTIPLICATION

PURPOSE:

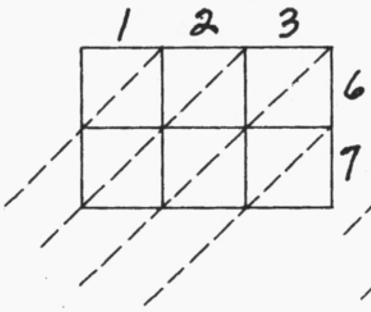
To learn how to draw boxes for multiplying two-digit numbers times three-digit numbers; to practice finding answers to multiplication problems using the boxes

MATERIALS:

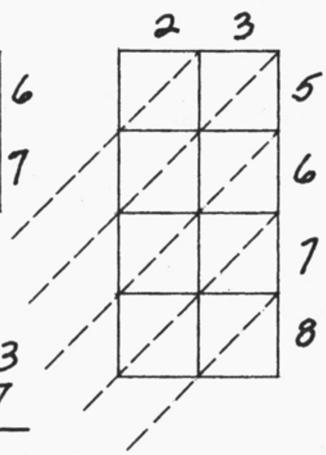
1. *Individual blackboards*
2. *Unlined paper*
3. *Dittoed copies of special multiplication matrix*

Any multiplication problem can be written in a box format. There are as many small squares along the top row of the box as there are numbers in the top row of a multiplication problem; likewise, for the bottom row and the squares on the side. Examples in this figure.

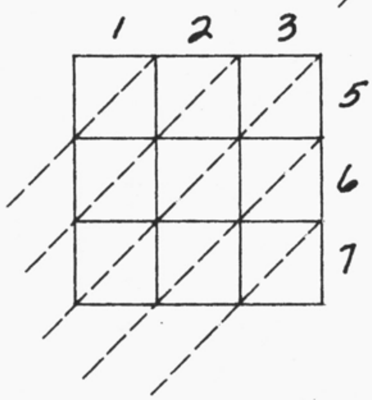
$$\begin{array}{r}123\\ \times 67\\ \hline\end{array}$$



$$\begin{array}{r}23\\ \times 5{,}678\\ \hline\end{array}$$



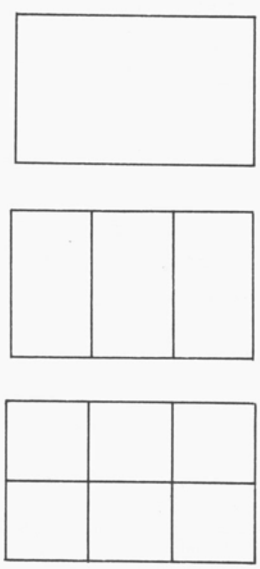
$$\begin{array}{r}123\\ \times 567\\ \hline\end{array}$$



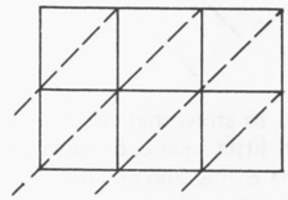
Teacher: There were 56 students and each had 123 orange chips. How many did they have altogether?
Student: A lot.
Teacher: True, but if I wanted to know exactly how many, how could I find out?
Student: We could get 56 students and give each 123 orange chips. Then we could count how many they had altogether.
Teacher: You're right. If we had enough students and enough chips we could work it out that way. We could also work it out using the boxes. Do you think the answers would be the same?
Student: Yes.
Teacher: Well, to save us the trouble of counting that many chips, I will show you how to draw a box to work this kind of problem. What numbers would I have to multiply on the boxes to get the answer?
Student: One hundred twenty-three times 56.

$$\begin{array}{r} 123 \\ \times\ 56 \\ \hline \end{array}$$

Teacher: I'll write that down. To draw a box for this problem I start with a rectangle. I want you to copy each of the steps on your blackboards. How many numbers in the top row?
Student: Three.
Teacher: Then I need to divide the rectangle so each of the three numbers will have its own space at the top. How many numbers in the second row?
Student: Two.
Teacher: Then I need two spaces on the side of the rectangle for the numbers in the second row. I've almost finished drawing the boxes. What's missing?



Student: The numbers.
Teacher: Yes. What must I add to make it look more like the boxes we've used before?
Student: Dotted lines?
Teacher: Yes. The same rule we used before for dotted lines still works. The dotted lines start at one corner of the square, cross diagonally to the opposite corner, and go out the other side.



Practice in drawing the box proceeds as in Lesson 9-11.

The teacher next constructs two-digit times three-digit problems and writes them on the overhead. For each problem, the teacher includes a brief description of a situation that might have produced the numbers. For example:

There are 24 classrooms in our school. If each had 325 books in it, how many books would be in the classrooms altogether?
There are 32 students in our room today. If I wanted each of you to have 250 cubes to work with, how many cubes would we need to have altogether?
There are 26 schools in our district. If each school had 725 students in it, how many students would that be altogether?

One or two key words are written by the numbers on the overhead as a reminder of what events might have produced them.

Students can learn to manipulate numbers without attributing any meaning to them, just as they can learn to read the words in a sentence without understanding what has been read. Words are used to convey thoughts; they lose their meaning if they are used only as the basis for making sounds. Numbers, too, have meaning. The students must have many opportunities to relate abstract numerals to reality, and vice-versa.

Each student computes the answers to as many problems as time permits. Near the end of the lesson the teacher points to the problems one at a time. Those students who have found an answer write it on their blackboards. The teacher scans the blackboards and selects a class answer. The process is repeated for each problem on the overhead.

Once the answers are written, the teacher and the class decide on a lable for each one. Do the numerals for an answer indicate how many students are in the room? Or how many Unifix cubes each has? Or something entirely different?

LESSON 9-13

LATTICE MULTIPLICATION

PURPOSE:

To practice box multiplication techniques while determining answers to selected multiplication problems

MATERIALS:

1. *Dittoed copies of the special multiplication matrix*
2. *Unlined paper*

The activities in this lesson are essentially the same as those in Lesson 9-11, related to examining the answers to series of problems for possible patterns. The only difference is these problems use a two-digit times three-digit multiplication box. An example of a possible sequence to explore can be seen in this figure.

$$\begin{array}{r}144\\ \times\ 12\\ \hline\end{array}\quad \begin{array}{r}143\\ \times\ 13\\ \hline\end{array}\quad \begin{array}{r}142\\ \times\ 14\\ \hline\end{array}\quad \begin{array}{r}141\ldots\\ \times\ 15\ldots\\ \hline\end{array}$$

LESSON 9-14

LATTICE MULTIPLICATION

PURPOSE:

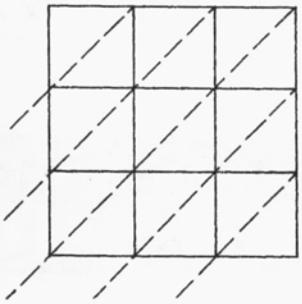
To learn how to draw boxes for multiplying three-digit numbers times three-digit numbers; to practice finding answers to multiplication problems using the boxes

MATERIALS:

1. *Individual blackboards*
2. *Unlined paper*
3. *Dittoed copies of special multiplication matrix*

The activities for this lesson are essentially the same as those in Lesson 9-12. Now, the boxes drawn and the problems written on the overhead are for problems involving a three-digit number multiplied by a three-digit number (the box needs to accomodate three numbers on a side).

$$\begin{array}{r}123\\ \times\ 456\\ \hline\end{array}$$



When the students have demonstrated their ability to draw boxes, problems are written on the overhead for them to solve. Each is accompanied by a description. For example:

There are 250 straws in a box. If each of the 376 students who eat lunch in the cafeteria used a box of straws a year, how many straws would be used up?

If each page in a book had 133 words on it, and there were 217 pages in the book, how many words would there be altogether in the book?

Class answers are then determined, recorded on the overhead, and labeled.

LESSON 9-15

LATTICE MULTIPLICATION

PURPOSE:

To practice box multiplication techniques while determining answers to selected multiplication problems

MATERIALS:

1. *Dittoed copies of the special multiplication matrix*
2. *Unlined paper*

This lesson is essentially the same as Lesson 9-11, examining the answers to series of problems for possible patterns.

Now the problems use a three-digit times three-digit multiplication box. An example of a possible sequence to explore may be seen in this figure.

$$\begin{array}{r} 123 \\ \times 124 \\ \hline \end{array} \quad \begin{array}{r} 122 \\ \times 125 \\ \hline \end{array} \quad \begin{array}{r} 121 \\ \times 126 \\ \hline \end{array} \quad \begin{array}{r} 120 \ldots \\ \times 127 \ldots \\ \hline \end{array}$$

LESSON 9-16

LATICE MULTIPLICATION

PURPOSE:

To practice finding answers to multiplication problems using the boxes

MATERIALS:

1. *Dittoed copies of special multiplication matrix*
2. *Unlined paper*

This lesson combines the activities in Lesson 9-9, 9-12, and 9-14, insofar as they relate to the teacher writing the problems for the students to work on the appropriate boxes. The teacher continues to write problems on the overhead while associating the numbers for those problems with real-life situations. However, the range encompasses all three problem sizes for which the students drew boxes in the earlier lessons. Here, they must decide which box to draw for which problem.

The problems for Lesson 9-9, 9-12, and 9-14 were all carefully selected to fit a certain box size. The problems now have no such restrictions, so the teacher may elect to allow students to help make up words to go along with them.

If this option is chosen, it is necessary to discuss with the class the nature of a multiplication problem. One student may suggest finding out how many books there would be altogether if each student had ten books in his or her desk. Another might ask how many books there would be altogether if one counted all the books in every student's desk. The first problem is multiplication, but the second is probably addition because it is unlikely that all students would have the same number of books in their desks.

Some problems are suitable for addition and multiplication. Are there problems for which a student can find an answer through multiplication and *not* addition? The teacher may lead the discussion, but the students must resolve the issues.

LESSON 9-17

LATTICE MULTIPLICATION

PURPOSE:

To practice box multiplication techniques while determining answers to selected multiplication problems

MATERIALS:

1. *Dittoed copies of the special multiplication matrix*
2. *Unlined paper*

This lesson combines the activities in Lessons 9-11, 9-13, and 9-15. The process of examining the answers found in series of selected problems for patterns continues. The range of problems encompasses all three sizes of boxes, and the students decide which boxes to draw for which problems.

LESSON 9-18

LATTICE MULTIPLICATION

PURPOSE:

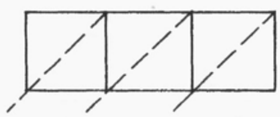
To examine specific kinds of multiplication problems for patterns

MATERIALS:

1. *Dittoed copies of the special multiplication matrix*
2. *Unlined paper*
3. *Individual blackboards*

Teacher: Someone please give me a number three digits long.

Student: Four hundred thirty-two.



Teacher: Here is the box you would use to multiply 432 times 1. Do it on your blackboards and show me what you get. . . . Most of you have 432 as the answer.

Now, multiply 432 times 10. You can decide what box to use for this one. Most of you have 4,320 on your blackboards. I'll make a record on the overhead of what we've done so far.

Now, multiply 432 times 100 and show me what you get. The most common answer I see is 43,200. I'll write that on the overhead, too.

$$432 \times 1 = 432$$

$$432 \times 10 = 4,320$$

$$432 \times 100 = 43,200$$

What do you think I will have you multiply 432 by now?

Student: One thousand.

Teacher: If you can figure out how to draw the box, multiply 432 by 1,000.

Can you look at the answers to the problems you've already done and predict the answer to 432 times 1,000? Do you think you would get a pattern for numbers other than 432 if you multiplied them first by 1, then by 10, then by 100, and then by 1,000? Would you still get a pattern if you used only two digits in the starting number instead of three? What would happen if the starting number had four digits? Can you find a number for which this 1, 10, 100, 1,000 pattern doesn't work?

Are the patterns for 2, 20, 200, and 2,000 the same as the ones for 1, 10, 100, and 1,000? Are they different? How?

What will happen if you try 3, 30, 300, and 3,000? Do you think you could predict the answer for 30 after only multiplying 3?

Students who wish to may spend the whole time thinking up new numbers to be multiplied by ones, tens, hundreds, and thousands. Others may choose to move from one series to the next. There is no prescribed order in which the number patterns must be explored.

This lesson may take only one class period or may absorb the students for several different sessions. If the students are fascinated by the patterns they see, this lesson may be presented again on subsequent days.

With the aid of the boxes and the special multiplication matrix, each student can now produce answers to multiplication problems whose products are greater than 100. The activities in many of the chapters that follow permit them to apply their skill in finding answers to multiplication problems in practical, problem-solving settings.