CHAPTER 10
Advanced Division

Prerequisite chapters:

Chapters 6 and 8

MATERIALS

For overhead projector:

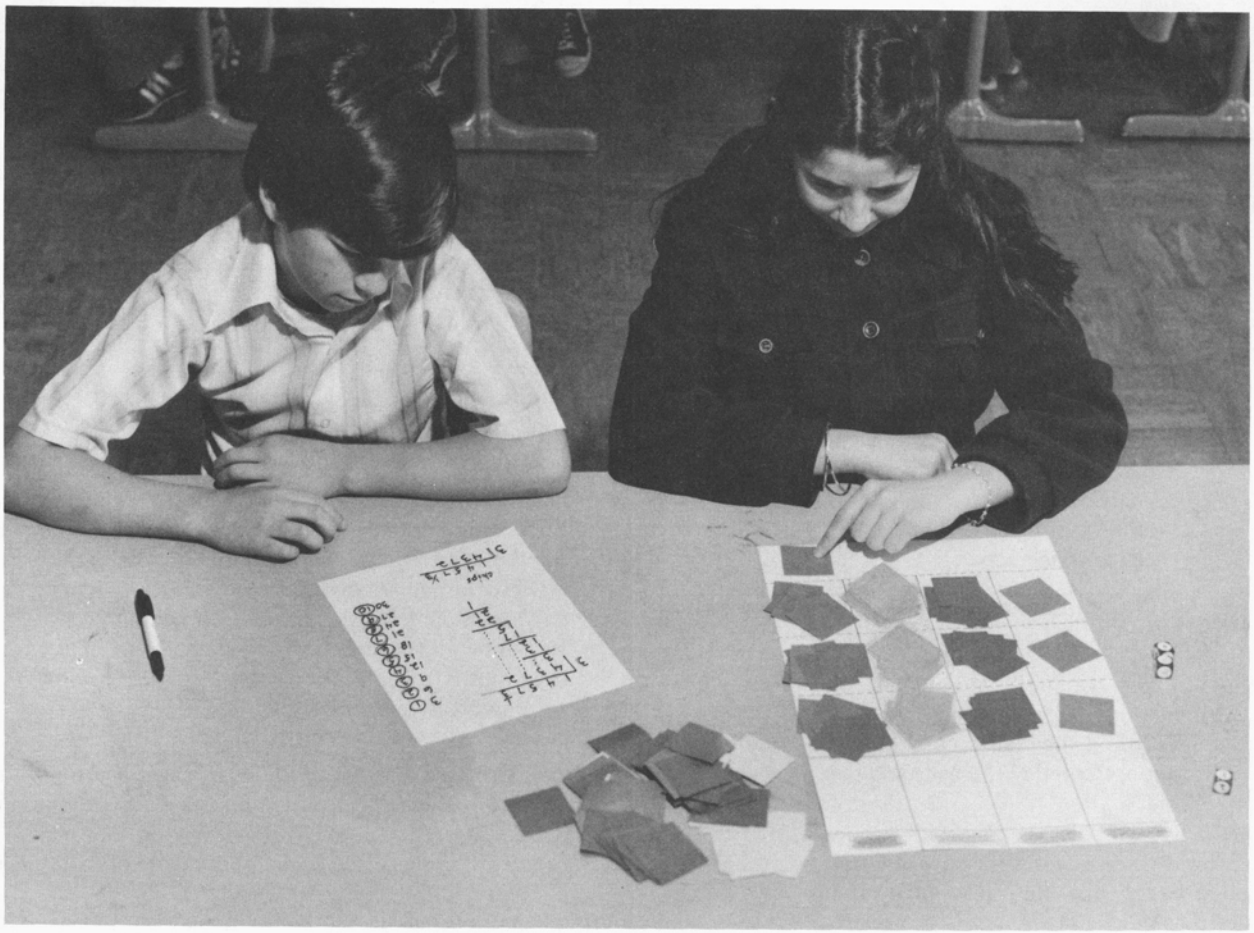
Transparencies 10 by 10 blank matrix Worksheet 1
Acetate squares with circles of three different sizes drawn on them (one per square) _____ Materials chapter, page 295
Acetate squares in five different colors _____ Materials chapter, page 297
Washable color marking pens _____ Materials chapter, page 297
Overhead projector dice _____ Materials chapter, page 296
Beans _____ Materials chapter, page 295

If no overhead projector is available:

Make charts in place of transparencies _____ Materials chapter, page 294
Circular cutouts of three different sizes _____ Materials chapter, page 295
Strips and squares of paper in five different colors _____ Materials chapter, page 297
Dice cards in bag _____ Materials chapter, page 296
Bean-shaped cutouts _____ Materials chapter, page 295

Student materials:

Dittos 10 by 10 blank matrix
Beans, cups, bowls _____ Materials chapter, page 295
Dice _____ Materials chapter, page 296
Individual blackboards _____ Materials chapter, page 294
Paper squares or chips in five different colors _____ Materials chapter, page 297
Crayons
Lined and unlined paper



In the activities for beginning division in Chapter 6, students used materials such as tiles to answer the relatively small division problems. As the problems increase in size, using materials soon becomes impractical.

This chapter presents students with an abstract system of long division that has been designed to minimize the potential for errors in calculation. Knowledge of why the system works is not emphasized. The emphasis is instead on providing evidence that it produces answers identical to those obtained by physically dividing objects.

The initial lessons acquaint students with use of their chip trading boards for answering division problems. Once this use is understood, they are used to verify answers obtained through the abstract system of long division.

LESSON 10-1

DIVISION WITH BEANS, CUPS, AND BOWLS

PURPOSE:

To test methods of dividing beans, cups, and bowls into groups of equal size

MATERIALS:

1. *If no overhead projector is available, bean-shaped cutouts*
2. *Clear acetate squares with circles of three different sizes drawn on them (one per square), or circular cutouts in three different sizes*
3. *Beans*
4. *Cups*
5. *Bowls*
6. *Individual blackboards*
7. *Unlined paper*

Chips on chip trading boards will be the primary tool used by the students to verify division answers. To insure students understand the division process represented by the chips, it is first presented with beans and cups.

Teacher: We will be working with groups of ten today. Please put three bowls, four cups, and five beans on your trading board, like this. How many beans do you have altogether?

Student: Five.

Teacher: How many beans are there altogether in the cups column?

Student: Forty.

Teacher: How many beans are there altogether in the bowls column?

Student: Three hundred.

Teacher: Whenever I ask "How many beans do you have altogether?" in this activity, I want to know how many beans you have altogether when you add the loose beans to the beans in the cups and bowls. How many beans do you have altogether?

Student: Three hundred forty-five.

Teacher: Now, I want you to divide the 345 beans into 5 equal piles. For each of the 5 piles, group the beans back into cups and bowls. Does anyone have a suggestion as to how we might divide the beans?

Student: Dump them all out of the bowls and cups. Divide them into five groups. Then take each group and see what it changes back into for beans and cups.

Teacher: Okay. In a minute we'll try it and see if that works. Can anyone think of a different way?

Student: Divide the beans into piles of five, then the cups, then the bowls.

Teacher: What would happen if you didn't have enough beans in the first column to divide into five piles?

Student: We do.

Teacher: That's true, but supposing I had told you to put out only four beans. Then what would you do?

Student: I could get some more beans out of the cup column.

Teacher: Okay. We'll try that way, too, in a minute and see if it works. Can anyone think of another way?

The teacher accepts all the students' suggestions; if no one suggests the following method, the teacher offers it.

Teacher: I'd like to suggest another way. We could divide each column into five piles, starting with the column having the most beans. For this problem, that would be the bowls column. We divide the bowls into five piles, and if any bowls are left over, we change them into cups. Then we divide all the cups into five piles, and change the leftover cups into beans. Any beans left over from the beans column is the remainder.

I'd like to know if this and the other suggested ways work. Try one of them or make up your own and see how many beans, cups, and bowls you get in your five piles. The people who suggested a method can explain it again if you wish.

If you finish using one method, you may try another.

LESSON 10-2

DIVISION WITH BEANS, CUPS, AND BOWLS

PURPOSE:

To retest methods of dividing beans, cups and bowls into groups of equal size

MATERIALS:

1. *If no overhead projector is available, bean-shaped cutouts*
2. *Clear acetate squares with circles of three different sizes drawn on them (one per square), or circular cutouts in three different sizes*
3. *Beans*
4. *Cups*
5. *Bowls*
6. *Individual blackboards*
7. *Unlined paper*

For this lesson each student uses all the suggested methods from Lesson 10-1 for dividing beans. Reworking one problem in several different ways helps students see an answer may be obtained through a variety of approaches.

Teacher: Who worked yesterday's division problem by dividing the beans into five piles then changing each back into bowls and cups?
Student: We did.
Teacher: Everyone put three bowls, four cups, and five beans back on your trading boards. After Ralph tells us how his group divided the beans, cups, and bowls, we'll all try it that way. If you have any questions, Ralph or someone who did it that way yesterday will help you.

The students present each system, including the teacher's, to the whole class and everyone tries it.

The teacher than poses the following questions:

Which method is easiest to understand?
Which method is fastest to use?
Why do all the different ways produce the same answer?

LESSON 10-3

DIVISION WITH BEANS, CUPS, AND BOWLS

PURPOSE:

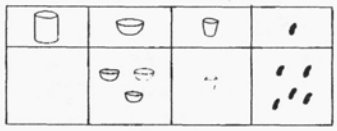
To create and solve division problems using beans, cups, bowls, and dice

MATERIALS:

1. *If no overhead projector is available, bean-shaped cutouts*
2. *Clear acetate squares with circles of three different sizes drawn on them (one per square), or circular cutouts in three different sizes*
3. *Beans*
4. *Cups*
5. *Bowls*
6. *Individual blackboards*
7. *Unlined paper*
8. *Overhead projector dice or dice cards in bag*
9. *Two dice per student: one numbered from zero to five, the other from zero to four*

In the preceding two lessons, the students practiced a variety of division techniques on a single problem. This lesson provides students with a method of creating their own division problems to increase their skill at dividing beans.

Teacher: Today, I want you to make up division problems on your trading boards by using dice. I will give you an example on the overhead. The first roll of my dice tells me how many beans to place in the beans column . . . what is the total of the two dice I rolled?
Student: Five.
Teacher: How many beans do I put on my trading boards?
Student: Five.
Teacher: Okay. The second roll tells me the number of cups . . . what is the total this time?
Student: One.
Teacher: How many cups do I put on my trading board?
Student: One.
Teacher: My third roll is for bowls. How many bowls do I put on my trading board?
Student: Three.



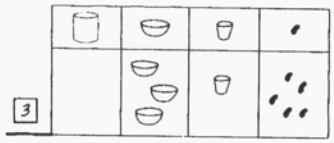
Teacher: I have three bowls, one cup, and five beans on my trading board. The next roll determines the number of groups I will divide the bowls, cups, and beans into. What is the total of the two dice?

Student: Three.

Teacher: How many piles will I divide all my bowls, cups, and beans into?

Student: Three.

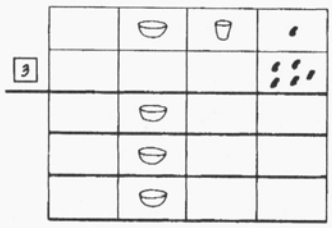
Teacher: I write the three here, so I can remember how many groups to divide everything into.



When it is your turn to divide the piles, you may use any system you wish. As I work *this* problem, though, I'll use the system that is easiest for me.

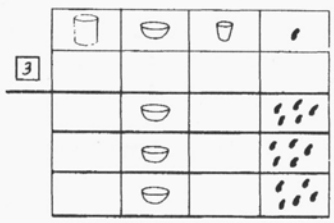
It helps me keep track of the three equal groups if I draw three rows underneath my trading board. If you want to do this, you could put the rows on another piece of paper. First I divide the bowls into three equal groups. How many bowls in each group?

Student: One.



Teacher: Now, I divide the cups into three equal groups . . . no, I can't do that, because I don't have enough cups. Okay, I'll dump all the beans out of this cup into the beans column.

Now, how many beans in each group?



Student: Five.

Teacher: For this problem, there were no beans left over. I don't know if that will happen with all our problems. If there are ever any beans left over, put them off to one side. They will be the remainder. How many bowls, cups, and beans do I have in each group?

Student: One bowl, zero cups, and five beans.

The students watch the teacher work the first example at the overhead. For the second example, the students tell the teacher what to do for each step. They then begin creating and working problems on their trading boards, throughout the available time.

Dividing by zero can present students with an interesting situation: how to divide something into no groups. Today, however, this is avoided by adding the rule that the bowls, cups, and beans are not to be divided into zero groups. If the total of the two dice is zero, they must be rolled again.

As the students work, the teacher walks around observing how well each student understands the process of creating and solving problems.

LESSON 10-4

DIVISION WITH BEANS, CUPS, AND BOWLS

PURPOSE:

To record division problems created using beans, cups, bowls, and dice

MATERIALS:

1. *If no overhead projector is available, bean-shaped cutouts*
2. *Clear acetate squares with circles of three different sizes drawn on them (one per square), or circular cutouts in three different sizes*
3. *Beans*
4. *Cups*
5. *Bowls*
6. *Individual blackboards*
7. *Unlined paper*
8. *Overhead projector dice or dice cards in bag*
9. *Two dice per student: one numbered from zero to five, the other from zero to four*

Teacher: The people I saw working division problems yesterday were doing a good job. But because I can't get around fast enough to see all your problems I want you to record your work on paper.

Division problems are usually recorded like this when you get to junior high or high school.

$$\begin{array}{r} 105\frac{1}{3} \\ \hline 3 \,|\, 316 \end{array}$$

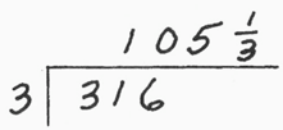
Write the number of bowls, cups, and beans you start with on the inside of the sideways *L*. On the outside, on the left, write the number for how many groups you divide all the bowls, cups, and beans into. On the top, write how many were in each equal pile. The number of left-over beans goes on top of the number for how many groups you divided the beans into.

The teacher quickly summarizes the entire recording method, then guides the students step by step through creating and recording a division problem.

The teacher rolls the dice to determine the number of beans, cups, and bowls to place on the overhead, and next, to determine the number of groups into which they are to be divided. The students record the numbers on their blackboards. They then tell the teacher how to work the problem, and record the answer. Once they understand the recording process they begin creating and recording their own division problems on paper.

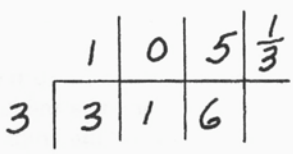
The most common difficulty is that students don't record the problem until they have finished it, and cannot always remember the number of bowls, cups, and beans with which they began. This difficulty is self-correcting since the problem cannot be recorded if only the answer is present.

The students' second most common recording difficulty is remembering to write the answer directly over the numbers for the bowls, cups, and beans. Once a student finds the answer, he or she is apt to write it without considering the position of the numbers. (An example can be seen in this figure.)



The teacher then cannot be sure if the problem has been worked correctly. When students record answers in this manner, the teacher writes a similar problem on the overhead and helps them discover for themselves the difficulty caused by an answer written in this form.

For students who have continued difficulty placing the answer over the appropriate numerals, the teacher presents a modified form of recording. These students may draw columns on their problems as a guide for the placement of each number.



The students create and record as many problems as time permits.

LESSON 10-5

DIVISION WITH BEANS, CUPS, AND BOWLS

PURPOSE:

To search for patterns in the answers to specific division problems

MATERIALS:

1. *If no overhead projector is available, bean-shaped cutouts*
2. *Clear acetate squares with circles of three different sizes drawn on them (one per square), or circular cutouts in three different sizes*
3. *Beans*
4. *Cups*
5. *Bowls*
6. *Unlined paper*
7. *Overhead projector dice or dice cards in bag*
8. *Two dice per student: one numbered from zero to five, the other from zero to four*

The students can now create division problems for beans, cups, and bowls. Before they transfer this knowledge to their chip trading boards, they explore the answers to selected division problems for patterns. These problems will be used as the basis for explorations in both chip trading and abstract long division. The common patterns found in the answers, regardless of the system of division, helps students see the sameness produced by beans, then chips, then numbers alone.

The first series of problems produces a pattern that, to adults, may seem self-evident. It must be remembered, however, that patterns are only obvious *after* they have been observed and understood.

Teacher: I want you to continue creating problems to divide. Today, though, divide all the problems you make up into groups of one. Record each problem on paper. As you work, look at the answers and find a pattern that will help you predict the answer to the next problems you make up for dividing by one.

Once the students have explored the answers obtained for dividing by one, they may pick any number as a divisor for their subsequent investigations. For students who need help deciding what divisor to try next, the teacher suggests twos, threes, fours, and so on.

All students will be able to see some patterns: that for dividing by one, for example. Some will see more. It isn't important that any particular pattern be seen; what is important is that they look and think about what they see.

LESSON 10-6

DIVISION WITH CHIPS

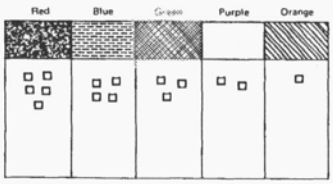
PURPOSE:

To test methods of dividing chips into groups of equal size

MATERIALS:

1. *Acetate squares or squares of colored paper in five different colors*
2. *Marking pens or strips of paper in five different colors*
3. *Paper squares or chips in five different colors*
4. *Crayons*
5. *Unlined paper*

The students are now ready to transfer their knowledge of bean division to dividing chips on a chip trading board—this expands the size of the problems thay may comfortably work with the aid of materials.



Teacher: Please put five red, four blue, three green, two purple, and one orange chip on your chip trading boards. I want you to divide all the chips into three equal piles. Does anyone have a suggestion as to how to divide them this way?

Student: We could trade all the chips in for orange chips and divide them into three piles. Then we could change the piles back into the other colors.

Teacher: That's one way we could try. In a little while we'll try that way and see if it works. Can anyone think of another?

Student: Start with the oranges. Divide them into three groups. If you haven't got enough oranges, change some purples into orange. Do that for each column.

Teacher: We'll try that way too and see if it works. Can anyone think of another way?

Student: Start with the reds. Divide them into three equal piles. Change any leftover reds into blues. Then divide up the blues, and change any leftovers into greens. Keep doing that for each column.

Teacher: We'll try that way, too. Can anyone think of another?

If no student suggests starting with the red chips (as the last student described), the teacher suggests that method.

Once a variety of ways have been suggested each student selects or makes up one way to try. They may ask a classmate to clarify his or her method. Students may work independently or in small groups.

Dividing chips is not the same as dividing beans. With beans, all the materials required to work the problem are actually present on the trading board. No matter which system of division was used, the basic operation involved redistributing beans that were already in the bowls and cups.

The division of chips cannot necessarily be accomplished by redistributing the materials on hand. For example, if students wish to convert all the chips to orange before the piles are divided, they will need 54, 321 orange chips. Once they appreciate the magnitude of that task, they quickly abandon it and switch to another system.

The students continue to test different methods as time permits.

LESSON 10-7

DIVISION WITH CHIPS

PURPOSE:

To retest methods of dividing chips on a chip trading board

MATERIALS:

1. *Acetate squares or squares of colored paper in five different colors*
2. *Marking pens or strips of paper in five different colors*
3. *Paper squares or chips in five different colors*
4. *Crayons*
5. *Unlined paper*
6. *Individual blackboards*

For this lesson the class discusses in turn each of the division methods attempted earlier. If a method was workable, each student uses it to divide the chips. If a method wasn't workable, the students who tried it share with the class the difficulties they experienced. This helps students see that although answers may be obtained through a variety of approaches, not all approaches are of equal merit.

When students have worked the problem using each successful method, they discuss the following questions:

Which method was the easiest to understand?
Which was the fastest to use?
Did all the different methods that produced answers get the same answer? Why? Why not?

LESSON 10-8

DIVISION WITH CHIPS

PURPOSE:

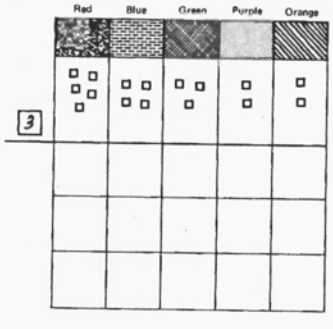
To create and solve division problems on a chip trading board

MATERIALS:

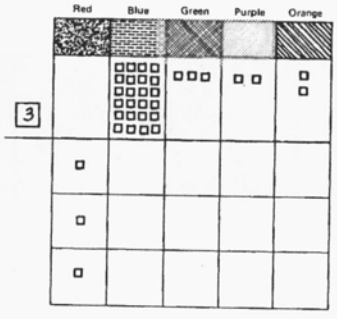
1. *Acetate squares or squares of colored paper in five different colors*
2. *Marking pens or, strips of paper in five different colors*
3. *Paper squares or chips in five different colors*
4. *Crayons*
5. *Unlined paper*
6. *Individual blackboards*
7. *Overhead projector dice, or dice cards in bag*
8. *Two dice per student: one numbered from zero to five, the other from zero to four*

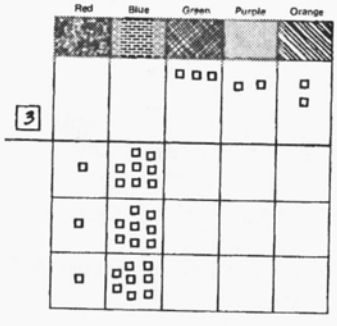
This lesson's activities are essentially the same as those in Lesson 10-3. Now, however, the materials are chips not beans, and the numbers to be divided contain five places as opposed to three.

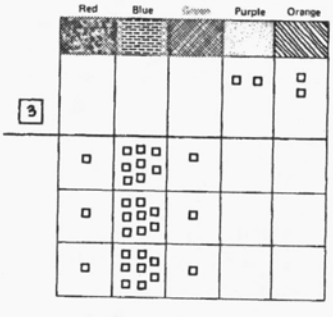
The teacher rolls dice to determine the number of chips to place in each column on the chip trading board, and the number of groups into which they are to be divided. To facilitate keeping track of the equal groups, the teacher draws an appropriate number of rows beneath the chip trading board.

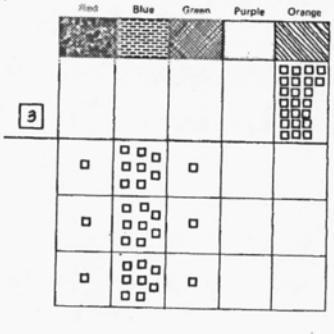


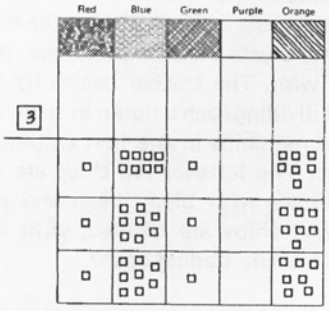
The teacher explains that the method selected to demonstrate the division process is the easiest for the teacher to use. When the students create division problems, they may use any system they wish. The teacher begins by starting with the red chips and dividing each column in turn, moving from left to right. The red chips in the first column are divided into three groups. The leftover red chips are converted to blues. This continues with blue, green and purple chips. Finally, the orange chips are divided, with any leftovers placed to the right of the trading board.











The lesson continues as in Lesson 10-3.

LESSON 10-9

DIVISION WITH CHIPS

PURPOSE:

To record division problems created on the chip trading boards

MATERIALS:

1. *Acetate squares or squares of colored paper in five different colors*
2. *Marking pens or strips of paper in five different colors*
3. *Paper squares or chips in five different colors*
4. *Crayons*
5. *Unlined paper*
6. *Individual blackboards*
7. *Overhead projector dice, or dice cards in bag*
8. *Two dice per student: one numbered from zero to five, the other from zero to four*

The activities are essentially the same as those in Lesson 10-4, but the materials are chips rather than beans, and the numbers recorded contain five places, not three.

LESSON 10-10

DIVISION WITH CHIPS

PURPOSE:

To search for patterns in the answers to specific division problems

MATERIALS:

1. *Acetate squares or squares of colored paper in five different colors*
2. *Marking pens or strips of paper in five different colors*
3. *Paper squares or chips in five different colors*
4. *Crayons*
5. *Unlined paper*
6. *Overhead projector dice, or dice cards in bag*
7. *Two dice per student: one numbered from zero to five, the other from zero to four*

The activities are essentially the same as those in Lesson 10-5, but the materials are chips rather than beans, and the numbers to be divided contain five places rather than three.

Once the students have examined the answers to dividing various amounts by one, they may pick any number as a divisor for subsequent investigations. They may also divide their chips into ten, twenty, thirty, or more groups. Dividing by larger amounts with chips is less unwieldy than using bowls, cups, and beans. Students may wish to take advantage of this and compare, say, the answers for dividing by two with those for dividing the same numbers by twenty.

The use of chips extends the range of patterns the students may explore. Division by two can easily be compared with division by twenty. Division by 200, however, presents problems of space.

Although materials provide students with the means of understanding an arithmetic operation, a time comes in both multiplication and division when the scope of a problem outstrips the practicality of using materials. In multiplication this situation was alleviated by lattice multiplication. To supplement the students' capabilities in division, the next series of lessons presents an abstract system of dividing that serves the same purpose as the boxes in multiplication.

LESSON 10-11

LONG DIVISION

PURPOSE:

To create and record repetitive addition patterns to be used in abstract division

MATERIALS:

1. *Individual blackboards*
2. *Lined paper*

Teacher: Write the numbers one through ten on your blackboards. Put a circle around each number like this.

I have a series of problems for you to work out. You may solve them using tiles, beans, or chips, or by working them in your head. All I want on your boards are the answers. Pick a number between one and ten.

Student: Five.

Teacher: Okay. Five is the number we will add for the first series of problems. What is five and zero?

Student: Five.

Teacher: The first answer goes by the one inside the circle. Now, add five to the first answer. What's five plus five?

Student: Ten.

Teacher: Ten goes by the two inside the circle. Add five to the second answer. Which number do I add the five to?

Student: Ten.

Teacher: What's five plus ten?

Student: Fifteen.

Teacher: Which number answer is 15?

Student: The third answer.

Teacher: Where do I write the third answer?

Student: By the three inside the circle.

Teacher: Okay . . . which numbers do I add now?

Student: Add 5 to 15.

Teacher: What will I get?

The students continue adding five to each successive answer until all ten answers have been recorded. The teacher then asks for a new number between one and ten and the process is repeated.

For the third example, the students write all ten answers in the series on their blackboards before showing their work to the teacher.

The first answer is always obtained by adding the selected number to zero. It is important for the students to keep track of the numbers for the answers in each series by writing them next to the circled numbers one through ten.

Once they understand how to add the same number to successive answers, they may begin creating their own problems, using any numbers they wish.

Teacher: When you have found ten answers for the number you select, hold up your blackboard so I can see it. I know a pattern for the numbers you are adding that lets me tell by looking at only two numbers if all the others are right. While I check your work, see if you can discover the pattern I use.

After the teacher has checked several boards using the unannounced pattern, the checking procedure changes.

Teacher: Now, we'll use another checking procedure. Don't show me your board, just tell me the number you started with and your tenth answer and I'll use my pattern to tell you if I think you added correctly.

Student: I started with 6 and ended with 60.

Teacher: I agree.

Student: I started with 12 and ended with 120.

Teacher: I agree.

Student: I started with 15 and ended with 145.

Teacher: Check again.

Student: I started with 15 and ended with 150.

Teacher: I agree.

After several student responses, the teacher begins a list on the overhead of the starting numbers, the tenth answers, and the teacher's opinion as to the accuracy of the addition.

Start	10th Number	Agree vs. Check Again
5	50	Agree
7	74	Check again
9	90	Agree
4	40	Agree
12	120	Agree
7	70	Agree
11	109	Check again
13	130	Agree
⋮	⋮	

The process is continued. Any student who, after checking a series of numbers again, feels the totals are correct despite the teacher's pattern, submits that problem to the whole class as a possible exception to the rule. This checking provides the individual both information on where an error in addition was made and evidence that it is difficult to find an exception to the teacher's pattern (there are no exceptions).

Although the teacher never announces the rule, students who think they see the pattern may share their discoveries with their classmates.

After the teacher has gone through the process about 15 times, the students are no longer told if their addition is correct. Instead, each is given a lined piece of paper and instructed to record as many ten-number series of answers as possible in the time available.

LESSON 10-12

LONG DIVISION

PURPOSE:

To record repetitive addition patterns on a matrix

MATERIALS:

1. *Blank matrix, ten squares by ten squares on a transparency, or blank matrix on a large tagboard*
2. *Dittoed copies of blank ten by ten matrix*
3. *Individual blackboards*

The numbers produced by the repetitive addition in the preceding lesson are multiplication patterns. To assist the students in making a connection between the series of numbers obtained by addition and the pattern they've observed in earlier lessons for multiplication, selected series of numbers are recorded on a matrix.

Teacher: Today I want you to do some problems like the ones you did yesterday and write the answers on a blank matrix.

Put the numbers one through ten down the side of your matrix. Now, circle each number. At the top of the first column, write a one. (See illustration, above right.)

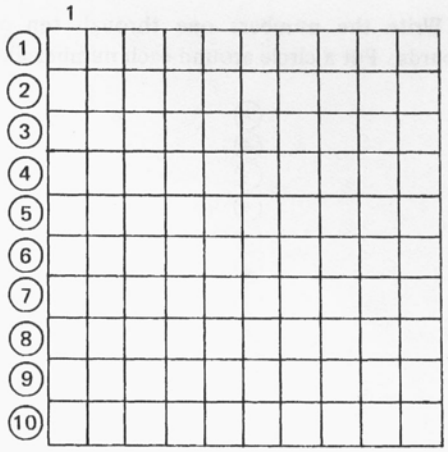
In this column, write the ten answers you get for adding one to each answer. What numbers do you add to get the first answer if you are going by ones?

Student: Zero and one.

Teacher: Then what's the first answer you would write on the matrix?

Student: One.

Teacher: What numbers do you add to get the second answer?



Student: One plus one.

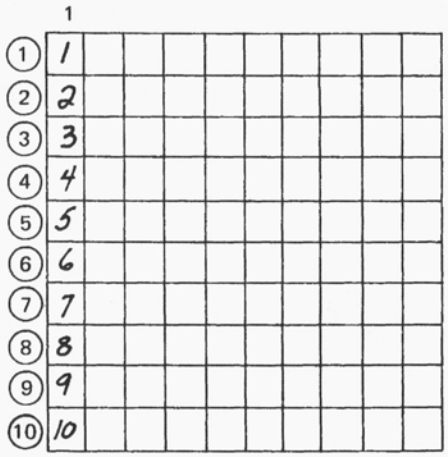
Teacher: What's the second answer?

Student: Two.

Teacher: And the third answer?

Student: Three.

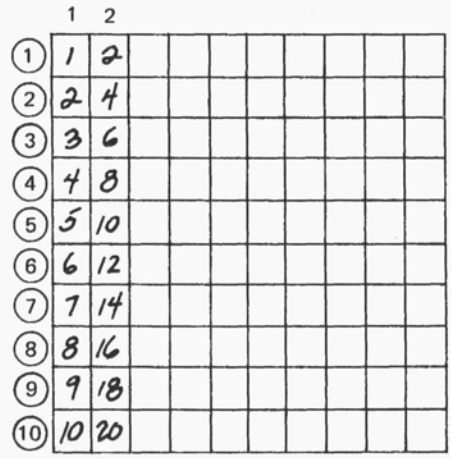
Teacher: Okay. On your matrix, write all the answers for adding one each time; then write them on your blackboard, too, so I can see what you got. Okay . . . I'll write what you have on my matrix too.



Now, at the top of the second column, write a two. What number do you think I'll have you add each time?

Student: Two.

Teacher: Write all the answers for adding twos on your matrix, then on your blackboards.



Okay . . . what number will I have you use next?
Student: Three.
Teacher: And after three?
Student: Four.
Teacher: Find all the answers for each new column of numbers and fill them in on your matrices. You don't have to write them on your blackboards yet.

As the students work, the teacher asks them to examine the rows and columns of numbers on their matrices for patterns that might help them predict answers. They next write the numbers for each column, one column at a time, on their blackboards. The teacher uses these numbers to fill in the matrix on the overhead and asks to examine it for patterns, as follows:

What patterns can you see as you look down the columns of numbers on the overhead?
Are there any patterns across the rows as well?
Are the patterns across the same as or different from the patterns going down? In what way? Why?
Have you seen any of these patterns before? Where? Why?

LESSON 10-13

LONG DIVISION

PURPOSE:

To learn an abstract system of division

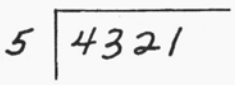
MATERIALS:

1. *Paper squares or chips in five different colors*
2. *Crayons*
3. *Individual blackboards*
4. *Unlined paper*

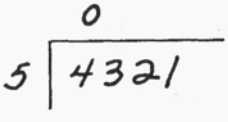
This lesson presents the students with an abstract system of dividing numbers. The system relies on the sets of ten numbers in a series the students learned to produce in the preceding two lessons. The use of the chip trading boards, however, provides the necessary bridge from the concrete knowledge gained from manipulating real objects to the abstract knowledge gained from manipulating numbers.

Teacher: Write on your blackboards the ten answers for starting with zero and adding fives. Make sure you put the answers next to the circled numbers one through ten. Then hold up your board so I can see what numbers to write on the overhead.

Now, we'll use these answers to get an answer to a division problem. I'll pick a division problem small enough so we can check the answer on our chip trading boards. I want to see if we get the same answer using the chip trading boards as we do using this special way of dividing.



Here is the problem. When we divided using chips, the fastest way for me to divide was to start with the chips in the left-hand column and divide each column one at a time across to the right. The method I will show you now starts the same way.
If these numbers were for chips on a chip trading board, which number would I divide first?
Student: Four.
Teacher: Four is the number I start with using this system, too. Look at the number in the list of fives-answers and see is there is any number you could subtract from four.
Student: Four.
Teacher: When you look at the list of fives-answers, only look at the numbers you wrote as the answers for going by five. Did you think I meant to look at the circled numbers, too?
Student: Yes.
Teacher: If you just look at the answers in the fives column, are there any numbers that you could subtract from four?
Student: No.



Teacher: Then above the four I write a zero. Since we couldn't take anything from the first number, we have to look at the first two numbers. Is there any number in the fives column we can take from 43?
Student: Fifteen.
Teacher: True. But what I meant to say was, what's the *largest* number on the fives list we could subtract from 43?
Student: Thirty-five.
Teacher: Any larger?

Student: Forty.
Teacher: Can I subtract 40 from 43?
Student: Yes.
Teacher: Any larger?
Student: Forty-five.
Teacher: Can I subtract 45 from 43?
Student: No.
Teacher: Then we can't use 45. We have to find the largest number on the list we *can* take from 43. What number would that be?
Student: Forty.

```
   0
5 | 4321
  -40
```

Teacher: Okay. We can use 40. Now, look at the number 40 in the fives column. What number answer was 40?
Student: What do you mean?
Teacher: What is the circled number next to the 40?
Student: Eight.
Teacher: The eight in the circle tells us that 40 is the eighth answer in the fives column. That eight goes above the three. Now, what is 40 from 43?

```
   08
5 | 4321
  -40
```

Student: Three.

```
   08
5 | 4321
  -40
    3
```

Teacher: Okay. We've used the four and the three. The next number we use is the two, by bringing it down beside the three, like this.

```
   08
5 | 4321
  -40 :
    32
```

Now, we have 32. What is the largest number in the fives column we can take from 32?
Student: Thirty.
Teacher: Can we take 30 from 32?
Student: Yes.
Teacher: Anything bigger?
Student: No.

```
   08
5 | 4321
  -40 :
    32
   -30
```

Teacher: Okay. We'll use 30. What number answer was that? What number is in the circle by the 30?
Student: Six.

```
   086
5 | 4321
  -40 :
    32
   -30
```

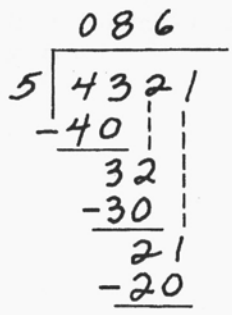
Teacher: Then the six goes in the answer row above the two. What's 30 from 32?
Student: Two.

```
   086
5 | 4321
  -40
    32
   -30
     2
```

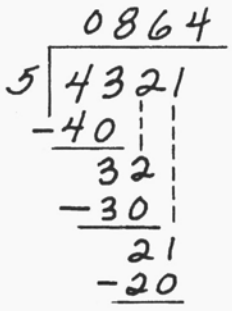
Teacher: Okay. We've used the four, the three, and the two. What do you think we will use next?
Student: The one.

```
   086
5 | 4321
  -40 : :
    32 :
   -30 :
     21
```

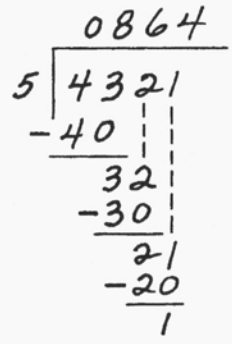
Teacher: The one is brought down beside the two, like this. What can we subtract from 21?
Student: Twenty.
Teacher: Anything larger?
Student: No.



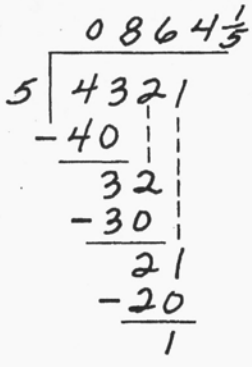
Teacher: What number do I write in the answer row?
Student: Four.



Teacher: Why?
Student: Because it's in the circle by the 20.
Teacher: What's 20 subtracted from 21?
Student: One.

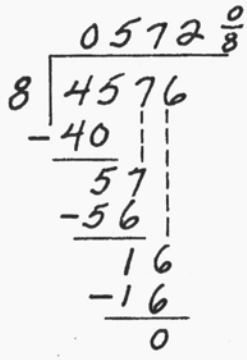


Teacher: We've used the four, the three, the two, and the one. Are there any numbers left to use?
Student: No.
Teacher: Can we take any numbers in the fives column from one?
Student: No.
Teacher: Then one is the remainder. The remainder is written like this.



Now the students check the answer obtained through the abstract system of division on their chip trading boards. In this case, they place four blue, three green, two purple, and one orange chip on their boards. The chips are then divided into five groups of equal size.

When the students have checked the accuracy of one problem, the teacher writes a second division problem on the overhead and the whole process is repeated. For the third example, the students answer the teacher's questions about the problem on their blackboards. The questions that would be asked for the sample problem of 4,576 divided by 8 are as follows:



What number would I use from the problem to get the ten answers?
Make a column on your blackboards for eight. Be sure to write each of the ten answers next to the appropriate circled number.
When the eights column is finished, what number do I look at first in the problem when I'm ready to begin dividing?
What is the largest number in the eights column I can subtract from four?
What number goes above four in the answer row?
What numbers do I look at next?
What is the largest number in the eights column I may subtract from 45?
What number do I write above the five in the answer row?
What is 40 from 45?
What number do I bring down?
What's the largest number from the eights column I can take from 57?
What number goes above the seven in the answer row?
What is 57 minus 56?
What number do I bring down next?
What number from the eights column may I subtract from 16?
What number would go above the six in the answer row?
What is 16 minus 16?
What is the remainder?
What is the answer to 4,576 divided by 8?

When the third example is completed the students verify the answer with their chip trading boards. The same series of questions are repeated for the fourth, and if time permits, fifth and sixth example.

LESSON 10-14

LONG DIVISION

PURPOSE:

To practice an abstract system of dividing numbers

MATERIALS:

1. *Individual blackboards*
2. *Chip trading boards and chips*

The students begin this lesson by instructing the teacher in the steps necessary to produce an answer, using the abstract system of division presented in the previous lesson.

$$4\overline{)1354}$$

Teacher: Here's a division problem. How do I work it?
Student: You have to make a fours column.
Teacher: Which four? And why?
Student: The four on the outside, because that's one of the rules you told us for dividing this way!
Teacher: How do I make a fours column?
Student: Write down the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10.
Teacher: Like this?

1, 2, 3, 4, 5, 6, 7, 8, 9, 10

Student: No, in a column, going down!
Teacher: Oh!
Student: Now put a circle around each number.
Teacher: Okay.
Student: By the one put a four, by the two put an eight, by the three put a twelve . . . by the ten put a forty.

4 (1)
8 (2)
12 (3)
16 (4)
20 (5)
24 (6)
28 (7)
32 (8)
36 (9)
40 (10)

Teacher: What next?
Student: Look at the one.
Teacher: Which one?
Student: The one in the problem, at the start.
Teacher: Okay. I'm looking.
Student: Put a zero above the one, in the answer row.
Teacher: Why would I want to do that?
Student: You can't take anything away from one.
Teacher: Can't I take *one* from one?
Student: Yes, but you don't have a one in the fours column.
Teacher: So?
Student: You can't take anything away from the one, from the fours column!

$$\begin{array}{r} 0 \\ 4\overline{)1354} \end{array}$$

Teacher: Okay. I've put a zero in the answer row. Now what?

The students continue directing the teacher, through two or three problems. Each student hears the process described by fellow students and sees the teacher carry out the directions. Not all students will learn the steps of this method by hearing them dictated. The ones who *do* learn in this matter, however, can tutor others.

The teacher now writes a problem on the overhead and the students attempt to work it on their blackboards. The teacher puts the divisor column on the overhead so the students' blackboards do not become overly crowded. Fast workers check the problem on their chip trading boards.

When most students have found an answer, they hold up their blackboards. The teacher scans them and selects the answer appearing most frequently as the class answer. Those who have the class answer on their boards describe how they arrived at that result as the teacher follows their instructions on the overhead. Students with different answers should find why they disagreed with the class answer.

When a substantial majority of students can work problems on their blackboards, they are ready to create and record division problems on their own, and assist those who cannot remember the steps involved.

This lesson provides practice in the steps necessary to produce answers using an abstract system of division. Once an answer is obtained abstractly, it is verified on the chip trading board. Students who have difficulty may use their boards *before* dividing problems abstractly. Knowing the answer in advance helps them concentrate on the sequence of steps in the abstract system.

The process is continued as time permits.

LESSON 10-15

LONG DIVISION

PURPOSE:

To create and solve division problems using an abstract system of dividing numbers; to check the answers on chip trading boards

MATERIALS:

1. *Chip trading boards and chips*
2. *One pair of dice per student: one numbered from zero to four, the second from zero to five*
3. *Unlined paper*

In this lesson, the students create and check their own division problems.

The technique of creating the problems is familiar. For each number to be divided, two dice are rolled and their numbers added together. Students then roll to determine the divisor. As was true for division with chips, no one may divide by zero. The only additional information students need is how many place numbers the dividend should be—four digits is a good place to start. Students who wish, may divide three or five place numbers to see what difference it makes in their answers.

After constructing problems with dice, students work them on paper using the abstract system of division. Problems are checked on the chip trading boards.

If a student's board results differ from the first answer, he or she asks another student to rework that problem. If both find the same discrepancy, they bring the problem to the teacher, who presents it to the class. Volunteers work out the problem to substantiate its validity as an exception or to find any steps that may have been left out by the first two students.

The fact that a system has been used to solve many problems does not prove it will work in all cases. Students should always look for exceptions, particularly in situations where they are using a system they were not taught for understanding. The search for exceptions can give them a pragmatic faith in the system's ability to produce correct answers.

The students create, solve, and check as many or few problems as time permits.

LESSON 10-16

LONG DIVISION

PURPOSE:

To create and solve division problems on the chip trading boards, to use the abstract system of division to check the answers

MATERIALS:

1. *Chip trading boards and chips*
2. *One pair of dice per student: one numbered from zero to four, the second from zero to five*
3. *Unlined paper*

The activities for this lesson are essentially the same as those for the previous lesson. However, the students create division problems on their chip trading boards, first, then verify the answers with the abstract system of division.

The only additional information the students need is how many columns on their boards to fill with chips: good columns to start with are the blue, green, purple, and orange. They may also use the red column, or delete the blue column to see what difference it makes in their answers.

LESSON 10-17

LONG DIVISION

PURPOSE:

To search for patterns in the answers to specific division problems

MATERIALS:

1. *One pair of dice per student: one numbered from zero to four, the other from zero to five*
2. *Unlined paper*

Teacher: When you were working on your chip trading boards, you found patterns for dividing that helped you predict some answers. What was the pattern you found for dividing by one?

Student: The answer was the same as the numbers we started with.

Teacher: What was the pattern for dividing by ten?

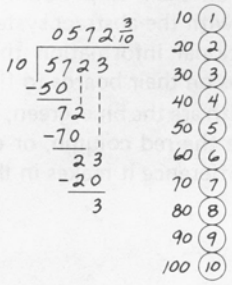
Student: All the numbers in the answer were moved over one column from the problem but they were still the same numbers.

Teacher: Okay. That was the chip trading pattern. Today I want you to make up problems for dividing by ten using numbers and no chips. See if the patterns you found using chips work when we only use numbers. Let me do an example first, to show you what I mean.

The teacher makes up a problem to be divided by ten and writes it on the overhead.

$$10 \overline{)\,5723}$$

Teacher: Look at the problem and tell me what answer you think I might get.
Student: 572 3/10.
Teacher: Okay. Let's work the problem using a tens column and see what we get.



The teacher works an example for dividing by ten on the overhead. The students then roll their dice for dividends for their own problems, each to be divided by ten. They first predict the answer, then work the problem.

Students who wish to, may explore the patterns for dividing by 20, 30, 40, 50, and so on, and compare them with those discovered earlier with chips. If the divisors for the current problems were also divisors for earlier chip trading problems, patterns found with chips can help the students predict the answer.

Students may explore various divisors for patterns: again dividing by 1, 2, or 3; dividing by 100, 200, 300, et cetera; and dividing the same number by 4, 40, 400, then 4,000.

LESSON 10-18

LONG DIVISION

PURPOSE:

To practice an abstract system of division while finding answers to teacher-created word problems

MATERIALS:

1. *Individual blackboards*
2. *Unlined paper*

Teacher: I have a word problem for you to solve. I want you to write the numbers for it on your blackboards.
We were having a party in class and I brought a bag full of peanuts to share. If there were 32 of us in the room and the bag held 125 peanuts, how many would we each get?
Okay. I see most of you have written this problem on your blackboards. What does the 32 stand for?

$$32 \overline{)\,125}$$

Student: How many of us there were.
Teacher: And the 125?
Student: How many peanuts were in the bag.
Teacher: I'll write a few words by this problem so we'll remember it when we come back to it later.

People $32 \overline{)\,125}$ Peanuts

Here's another problem: There are, 657 paperback books in the class library. How many books could we each have if there were only twenty of us in the room?
Write the problem on your blackboards.

The teacher creates about ten word problems and the students compute the answers for as many as time permits. Then, the teacher points to the problems one at a time; students who have found an answer for that problem write it on their blackboards. The teacher scans the boards and selects a class answer. The process is repeated until there is a class answer for each problem.

The teacher next asks the students what the answer means. The answer to the problem for people and peanuts is 3 29/32. Does this number tell how many people were in class? How many peanuts each person got? Or how many peanuts were in the bag?

Students must know how to identify the numbers in a word problem to set up and solve it. For this skill to be of real value they must also know what the answer means.

LESSON 10-19

LONG DIVISION

PURPOSE:

To practice an abstract system of division while finding answers to student-created word problems

MATERIALS:

1. *Individual blackboards*
2. *Unlined paper*

This lesson's activities are essentially the same as those in the previous lesson. The only difference is the problems written on the overhead come from situations described by the students rather than the teacher. As the students suggest potential division problems, the teacher and the class discuss what is and isn't an appropriate situation for dividing. Knowing how many pieces of sugarless gum each person should receive if the amount of gum available were distributed evenly can be found by division. Knowing how many pieces each person would get if each one in turn took a handful cannot.

With the aid of the abstract system of division to which they have been introduced, each student can now produce answers to division problems that are impractical to solve using materials. The activities in many of the following chapters permit students to apply this skill in practical, problem-solving settings.